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THE EDITOR'S PAGE

QUO VADIS?

From time to time we receive questions from our readers concerning the content and purposes of the MATHEMATICS MAGAZINE. So it seems worth while to explain the purpose of our publication.

The MATHEMATICS MAGAZINE is a journal devoted to collegiate mathematics. We are striving to maintain a level of content between the AMERICAN MATHEMATICAL MONTHLY and the MATHEMATICS TEACHER. Thus, although we publish some research papers, we are not primarily a research journal. And, although we have a department devoted to teaching of mathematics, this is not our main function. We are trying to publish a wide variety of mathematical material that will be of interest to the college mathematics professor, the advanced undergraduate student, and the teacher of secondary school mathematics.

In order to achieve this purpose we publish articles showing new insights into well-known problems, hueristic treatments of both elementary and advanced mathematical topics, discussions of methods of teaching mathematics, and reviews of current mathematical publications. Our Problems and Questions department attempts to propose problems at a wide variety of levels of difficulty. We do, in fact, have contributors to this department ranging from well-known research mathematicians to gifted high school students.

We are encouraged to continue to aim for this level of mathematical exposition. As one of our readers put it recently in a letter to us, "There is a distinct need for a periodical at this level, and I hope you will not let the level creep up. Other periodicals can take care of doctor's dissertations and up-to-the-minute research. We need interesting material for the large group of not-too-sophisticated mathematicians."

The editors of the MATHEMATICS MAGAZINE face a continuing problem in trying to achieve our objective. A magazine can only print the articles which it receives. In order to maintain the desired level of content, we need to receive articles written at that level. So I will finish with an appeal to all those who are interested in writing about mathematics. We need *expository* articles about mathematics at all levels, from topics in secondary school mathematics to advanced modern topics. We would welcome articles on the teaching of all levels of mathematics. Who would discuss in our pages the impact on college curricula of the introduction of elementary set theory in high schools? Do you have a new point of view concerning some element of mathematical philosophy? Has everything interesting about the history of mathematics been said? Do you have a novel proof of a well-known proposition?

If we can receive articles of this nature, then the MATHEMATICS MAGAZINE can continue to achieve its chosen objective.

R.E.H.

MATHEMATICAL ANALYSIS OF THE PARALLEL PARKING PROBLEM

William A. Allen

Almost every driver has had the experience of finding his automobile wedged between two other parked cars. A good driver can usually extricate his vehicle by performing an appropriate sequence of operations in an intuitive manner. The admissible operations include moving his car back

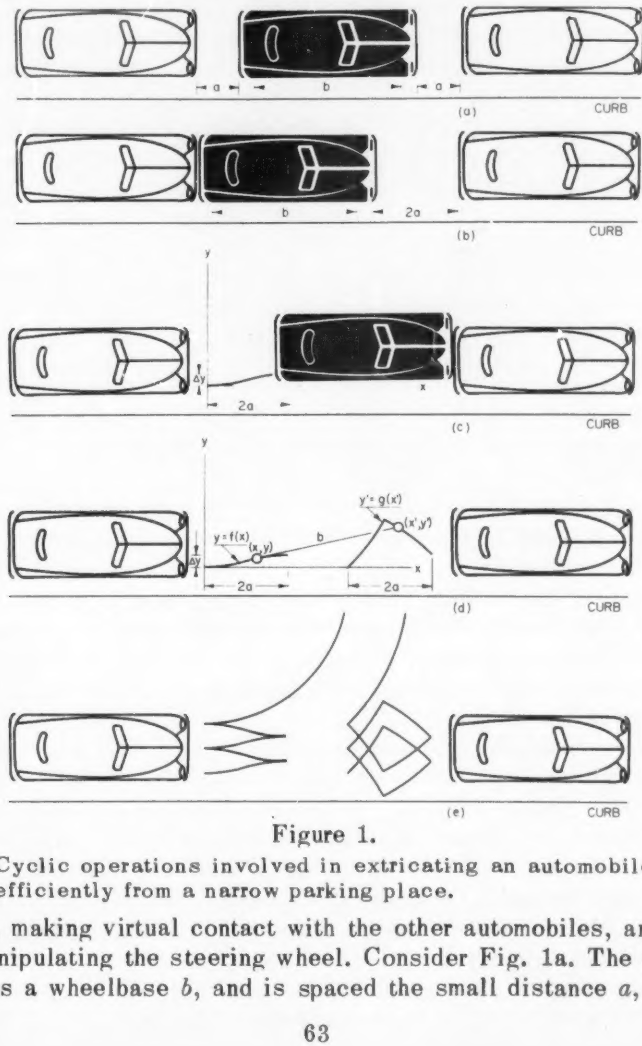

Figure 1.
Cyclic operations involved in extricating an automobile efficiently from a narrow parking place.

and forth, making virtual contact with the other automobiles, and concurrently manipulating the steering wheel. Consider Fig. 1a. The car in the middle has a wheelbase b, and is spaced the small distance a, both from

the car in front and the car in the rear. Assume that the driver backs his car until his rear bumper virtually touches the front bumper of the car behind him. Figure 1b is assumed to be the initial configuration of the parallel parking problem which can be formulated as follows: *What cyclic operations must the driver perform in order to displace his automobile laterally sufficiently far to escape confinement from a narrow parallel parking place?*

If the confinement is close, the trapped automobile can only oscillate back and forth, and any lateral motion of the car will be small for each cycle. Since the car initially is parallel to the curb, the imposition of periodicity implies that the car becomes parallel to the curb again at the end of each cycle of operation. Consider a cartesian coordinate system with the origin determined by the initial position of the right rear wheel as shown in Fig. 1c, and let the x axis be directed parallel to the curb. Assume, Fig. 1d, that the initial cycle of operation produces a track of the right rear wheel specified by the relation

$$y = f(x) \,, \tag{1}$$

with the boundary conditions

$$\begin{gathered} y(0) = 0 \,, \quad y(2a) = \Delta y \,, \\ \dot{y}(0) = \dot{y}(2a) = 0 \,, \end{gathered} \tag{2}$$

where the dots imply differentiation with respect to x. During the initial cycle of operation, the car is displaced the distance Δy laterally. It is sufficient to consider only the initial cycle since the second, or reverse cycle of operation, will be a mirror image of the first.

The track of the right rear wheel determines the track of the right front wheel. Erect a tangent at point (x, y) on the curve $y = f(x)$ and construct on the tangent the distance b to locate the point (x', y'); the locus of all such points is assumed to specify the track, $y' = g(x')$, of the right front wheel. The track of the right front wheel can be written in parametric notation

$$x' = x + b(1+\dot{y}^2)^{-\frac{1}{2}} \,, \tag{3}$$

$$y' = y + b\dot{y}(1+\dot{y}^2)^{-\frac{1}{2}} \,. \tag{4}$$

Consider the integral

$$W = \int y'dx' . \tag{5}$$

Substitute Eqs. (3) and (4) into Eq. (5) to obtain

$$W = \int y dx - \int \frac{by\dot{y}d\dot{y}}{(1+\dot{y}^2)^{3/2}} + \int \frac{b\dot{y}dx}{(1+\dot{y}^2)^{\frac{1}{2}}} - \int \frac{b^2\dot{y}^2 d\dot{y}}{(1+\dot{y}^2)^2} \,. \tag{6}$$

Integrate the second term on the right-hand side of Eq. (6) by parts to obtain

$$-\frac{b}{2}\int\frac{2y\dot{y}d\dot{y}}{(1+\dot{y}^2)^{3/2}} = -\frac{b}{2}\left[-\frac{2y}{(1+\dot{y}^2)^{1/2}}+\int\frac{2\dot{y}dx}{(1+\dot{y}^2)^{1/2}}\right]. \tag{7}$$

Substitute Eq. (7) into Eq. (6) to obtain

$$\int y'dx' - \int ydx = \frac{by}{(1+\dot{y}^2)^{1/2}} - b^2\int\frac{\dot{y}^2d\dot{y}}{(1+\dot{y}^2)^2}. \tag{8}$$

Integrate Eq. (8) over a complete cycle. Imposition of the boundary conditions, Eqs. (2), reduces Eq. (8) to

$$\int y'dx' - \int ydx = b\Delta y. \tag{9}$$

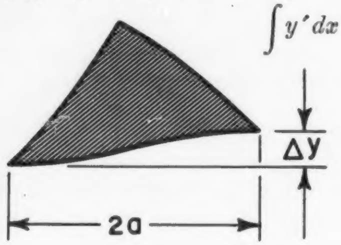

Figure 2
Area enclosed by tracks of right front and rear wheels. The lateral distance Δy is a maximum whenever the enclosed area is maximum.

Figure 2 is a sketch illustrating the physical interpretation of the integrals of Eq. (9). The top curve of Fig. 2 represents the track of the right front wheel, the bottom curve represents the track of the right rear wheel. The area enclosed by the two curves, from Eq. (9), is a direct measure of the distance Δy that the car is displaced laterally in one cycle. In order to maximize the distance Δy that the car moves laterally for each cycle of operation, the area shown in Fig. 2 must be a maximum.

Consider the trajectory of the right rear wheel that results in maximum displacement Δy. The lengths r_1 and r_2 in Fig. 3 are the respective minimum left turning radius and right turning radius of the right rear wheel. If

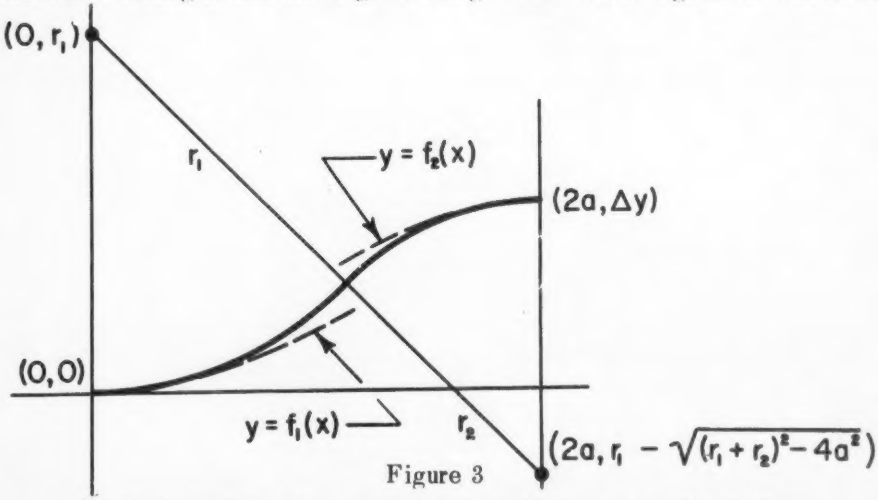

Figure 3
Trajectory of the right rear wheel treated as arcs of a circle.

t is the *tread* of the car, then

(10) $$r_1 = r_2 + t \ .$$

The dashed lines in Fig. 3 represent arbitrary curves $y_1(x)$ and $y_2(x)$ with the respective bounded curvatures $1/\rho_1 \leqq 1/r_1$ and $1/\rho_2 \leqq 1/r_2$. Each of the dashed lines lies either on or outside of the circle determined by its associated arc. This fact is inferred by considering the arcs as the limits of the arbitrary curves as their respective curvatures approach $1/r_1$ and $1/r_2$. As $1/\rho_1 \to 1/r_1$, for example, the points on $y_1(x)$ move counterclockwise with reference to the origin and toward the circular arc. Thus, Δy is maximum when $y_1(x)$ and $y_2(x)$ are circular arcs. In this case

(11) $$\Delta y = (r_1 + r_2) - [(r_1 + r_2)^2 - 4a^2]^{1/2} \ .$$

For the special case where Δy is maximum the area of Fig. 2 is determined by circular arcs of radii r_1, r_2, $(r_1^2 + b^2)^{1/2}$, and $(r_2^2 + b^2)^{1/2}$. Each wheel turns through the same arc; that is, $\sin^{-1} 2a/(r_1 + r_2)$. At any instant of time all four wheels are rotating around a common point. It can be verified by integration, or otherwise, that the area of Fig. 2 is given by $b\Delta y$ where Δy is specified by Eq. (11).

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FINITE SURFACES
A STUDY OF FINITE 2-COMPLEXES

E. F. Whittlesey

Part II. The Canonical Form

We give now an algorithm by which the system of words for a complex can be reduced to a canonical form. Examples are given after the algorithm.
Step 1. *Collect all the singular polygons on a single surface component into a single polygon.*

– namely, by the composition rule for polygons.

Having, at this juncture, a system of words such that each word represents exactly one surface component, we can proceed to the canonical form now without (explicit) use of cutting and pasting of polygons by means of the *Circulation rules. The following substitutions of one word for another are admissible if x is a regular edge.*

Rule 1a. $\cdots xQ\beta T\cdots xQ\cdots \longleftrightarrow \cdots xT\cdots xT\beta^{-1}Q\cdots$

1b. $\cdots T\beta Px\cdots Px\cdots \longleftrightarrow \cdots Tx\cdots P\beta^{-1}Tx\cdots$

2a. $\cdots xP\alpha T\cdots Px^{-1}\cdots \longleftrightarrow \cdots xT\cdots P\alpha Tx^{-1}\cdots$

2b. $\cdots P\beta Tx\cdots x^{-1}T\cdots \longleftrightarrow \cdots Px\cdots x^{-1}P\beta T\cdots$

Proof. 1b follows from 1a by interchange of members and inversion of both. 2b follows from 2a by cyclic permutation. The proof of 1a is given and illustrated below, the proof of 2a is similar and is omitted.

$\cdots xQ\beta T\cdots xQ\cdots$

partition $\downarrow$

composition $\uparrow$

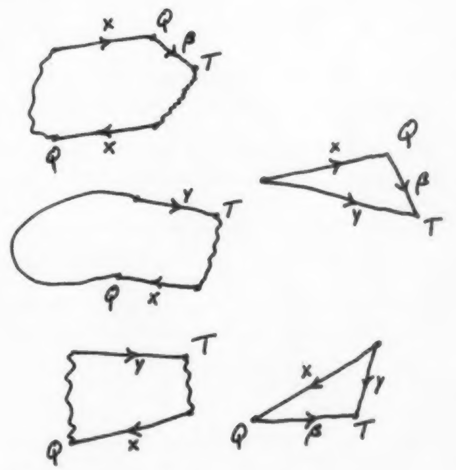
$\cdots yT\cdots xQ\cdots \quad xQ\beta Ty^{-1}$

$\updownarrow$ identity $\quad \updownarrow$ inversion

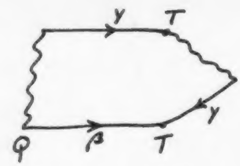
$\cdots yT\cdots xQ\cdots \quad yT\beta^{-1}Qx^{-1}$

composition $\downarrow$

partition $\uparrow$

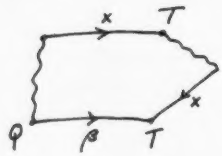


$\cdots yT \cdots yT\beta^{-1}Q\cdots$



substitution

$\cdots xT \cdots xT\beta^{-1}Q\cdots$



Step 2. *Collect the crosscaps.*

Suppose we have a word $\cdots Px\cdots Px\cdots$ with x a regular edge. Apply the first rule to bring the x's together: $PxPxP\cdots$. Then set $x = yTz$, get: $PyTzPyTzP\cdots$. Collect the "crosscap" $z\cdots z$, get $PyTzTzTy^{-1}P\cdots$ or, permuting, $TzTzTy^{-1}P\cdots PyT$. T is regular. If the sequence of dots is a block containing further crosscaps, collect as before, then use the second rule to put in front of the y^{-1}:

$$TzTzTy^{-1}\cdots Mv\cdots Mv\cdots yT$$

(first rule)

$$TzTzTy^{-1}\cdots MvMvM\cdots yT$$

(second rule)

$$TzTzTy^{-1}MvMvM\cdots yT$$

(first rule, twice)

$$TzTzTvTvTy^{-1}M\cdots MyT.$$

T is still regular, being inarticulate and adjacent to regular edges only. Finally, we have the form: $Aa_1Aa_1A\cdots Aa_qAa_qAd\cdots d^{-1}A$, where A is regular, d is regular, and the sequence of dots between the d's represents a block containing no further crosscaps. (It may be, of course, that q is 0.)

Step 3. *Collect the handles.*

Suppose we have, in some word, $\underset{1}{\cdots}a\underset{2}{\cdots}b\underset{3}{\cdots}a^{-1}\underset{4}{\cdots}b^{-1}\underset{5}{\cdots}$ where a and b are regular. Such a subsequence we call a *handle*. Suppose q, above, is > 0. "Collect" the handle as follows:

$$\alpha A\beta PaR\gamma SbV\delta Ra^{-1}P\epsilon Vb^{-1}S\zeta, \quad \text{where} \quad \alpha = Aa_1Aa_1\cdots Aa_qAa_q.$$

$$\downarrow$$

$$\alpha AaR\gamma SbV\delta Ra^{-1}A\beta P\epsilon Vb^{-1}S\zeta$$

$$\downarrow$$

$$\propto AaR\gamma SbA\beta P\epsilon V\delta Ra^{-1}Ab^{-1}S\zeta$$

$$\downarrow$$

$$\propto AaA\beta P\epsilon V\delta R\gamma SbAa^{-1}Ab^{-1}S\zeta$$

$$\downarrow$$

$$\propto AaAbAa^{-1}Ab^{-1}A\beta P\epsilon V\delta R\gamma S\zeta\,.$$

The collection process is summarized by the permutation symbol 14325. In the same fashion, we can collect any further handles and move them forward. We get, finally,

(*) $Aa_1Aa_1\cdots Aa_qAa_qAb_1Ac_1Ab_1^{-1}Ac_1^{-1}A\cdots Ab_sAc_sAb_s^{-1}Ac_s^{-1}Ad\gamma d^{-1}A$,

where A, d are regular, γ contains no crosscaps (rule 2 does not invert) and no handles. If q is 0, we can replace the initial vertex, B say, by an incision:

$$B\cdots B \longrightarrow Bd^{-1}AdB\cdots B \longrightarrow AdB\cdots Bd^{-1}A \quad \text{(by permuting).}$$

Thus, as before, we can begin with the assumption that the initial vertex is regular. Even if there are no crosscaps or handles, we can, as above, put the word in the form $Ad\cdots d^{-1}A$, where A and d are regular. Thus we can assume finally that each word is in the form (*) above, where q or s may be 0.

Step 4. *If $q > 0$ and $s > 0$, convert each handle into two crosscaps.*
Thus:

$$\cdots aabcb^{-1}c^{-1}\cdots$$

$$\downarrow$$

$$\cdots ac^{-1}b^{-1}ab^{-1}c^{-1}\cdots$$

$$\downarrow$$

$$\cdots aba^{-1}bc^{-1}c^{-1}\cdots$$

$$\downarrow$$

$$\cdots aabbc^{-1}c^{-1}\cdots$$

The rest of the word is quite unchanged. Thus each word is now in one of the forms

$$\propto d\beta d^{-1}A$$

where

$$\propto \; = \begin{cases} A \\ Aa_1Aa_1\cdots Aa_qAa_qA & \text{(mutually exclusive cases)} \\ Aa_1Ab_1Aa_1^{-1}Ab_1^{-1}A\cdots Aa_pAb_pAa_p^{-1}Ab_p^{-1}A \end{cases}$$

and where the a's, b's, d and A are regular, and it may be that the entire word consists of $\propto$ alone, there being no final $d\beta d^{-1}$, β contains no cross-caps or handles.

Step 5. *Separate the cuffs.*

Consider the final block $Ad\beta d^{-1}A$. If any regular edge, r say, occurs in β, then it occurs again in β as r^{-1}: $\beta = \cdots r\gamma r^{-1}\cdots$. Again, γ may contain such a sequence: $\gamma = \cdots k\delta k^{-1}\cdots$. But the process must end and we reach an innermost sequence of this sort. Move it forward, as follows, by the second circulation rule:

$$\cdots Ad\cdots t\epsilon t^{-1}\cdots d^{-1}A$$

$$\downarrow$$

$$\cdots At\epsilon t^{-1}Ad\cdots d^{-1}A\ .$$

Continue with the block between the d's, get finally

$$\propto d_1\propto_1 d_1^{-1}Ad_2\propto_2 d_2^{-1}A\cdots Ad_k\propto_k d_k^{-1}A$$

where $\propto$ is as before, A and d_i are regular, $\propto_i$ contains no regular edge (although possibly a regular vertex). Notice that $\propto_i$ is either a vertex, or is a sequence of line-singular edges and in this case can be replaced by a cyclic permute of itself, by the second circulation rule.

Step 6. *Remove all incisions; if the entire system is the sphere, $AdPd^{-1}A$, cancel the incision, but keep the single vertex A; similarly if there are several disjoint spheres.*

– by the incision rule. Bear in mind that if P above is singular, then $AdPd^{-1}A$ is not an incision and cannot be cancelled. Any incision occurs as a terminal "cuff", dPd^{-1}, P regular, and we look for and remove them.

Step 7. *Compose all lines.*

This means that we are to use the edge composition rule on the centers, $\propto_i$, of the terminal cuffs, until it can no longer be applied, no matter what cyclic permute of itself we substitute for any $\propto_i$.

The system is now in canonical form. Notice that the last step puts the singular graph in canonical form (in the sense of canonical form for 1-complexes). Each word is in the general form $\propto\beta$, where

$$\beta = d_1\propto_1 d_1^{-1}\cdots d_r\propto_r d_r^{-1}$$

$$\alpha = \begin{cases} A \\ a_1 a_1 \cdots a_q a_q \\ a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_p b_p a_p^{-1} b_p^{-1}, \end{cases}$$

the forms possible for α are mutually exclusive, and β may be empty.

It is obvious that

(1) *the order of the words in the canonical form is immaterial,* it may be altered arbitrarily.

(2) *The order of the terminal cuffs on any word may be changed arbitrarily.* This follows from the second circulation rule.

(3) *The letters in any* α_i *may be permuted cyclically.* This follows from the second circulation rule.

(4) *Any edge may be reöriented* (this means that for any edge, b say, b and b^{-1} may be interchanged throughout the system). This is a rule of formal equivalence.

(5) β *can be replaced by* β^{-1}. For we can replace $\alpha\beta$ by $\beta^{-1}\alpha^{-1}$, and this, by cyclic permutation, may be replaced by $\alpha^{-1}\beta^{-1}$, and α^{-1} is of the same general form as α, and β^{-1} is of the same general form as β.

If α consists of q crosscaps, $q > 0$, the surface component is called *non-orientable,* otherwise it is *orientable.*

(6) *A terminal cuff on any non-orientable surface component may be inverted.*

Proof. Given: $\cdots aad_1\alpha_1 d_1^{-1}\cdots d_i\alpha_i d_i^{-1}\cdots$, we can, by the second circulation rule, move the crosscap aa past each cuff until it precedes $d_i\alpha_i d_i^{-1}$:

$$\cdots d_1\alpha_1 d_1^{-1}\cdots aad_i\alpha_i d_i^{-1}\cdots .$$

Then by the first circulation rule, get

$$\cdots d_1\alpha_1 d_1^{-1}\cdots ad_i\alpha_i^{-1} d_i^{-1} a\cdots .$$

By the second circulation rule, get

$$\cdots d_1\alpha_1 d_1^{-1}\cdots d_i\alpha_i^{-1} d_i^{-1} aa\cdots ,$$

and then return the crosscap again:

$$\cdots aad_1\alpha_1 d_1^{-1}\cdots d_i\alpha_i^{-1} d_i^{-1}\cdots .$$

(1) to (6) are called *trivial isomorphisms* of the canonical form. They have important topological significance and interpretation in cases (2), (3), (5), and (6) to which we shall return later.

Examples of canonical form and reduction thereto.

(1) The sphere: A

(2) The pinched sphere: from an earlier example we already have

$$AaAa^{-1}A$$

troduce an incision: $AbBb^{-1}AaAa^{-1}A$

ermutation) $Bb^{-1}AaAa^{-1}AbB$

ıen by the second circulation rule:

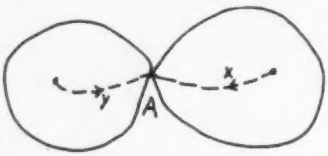
$$BaAa^{-1}BbAb^{-1}B\ ,$$

simply,

$$aAa^{-1}bAb^{-1}$$

.he canonical form.

(3) Two tangent spheres:



$$\left.\begin{matrix} xAx^{-1} \\ yAy^{-1} \end{matrix}\right\} \text{ – the canonical form.}$$

(4) n spheres with "north poles" identified and "south poles" identified (an onion). From an earlier example we already have

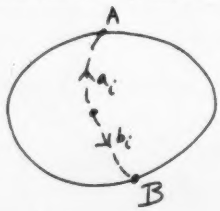
$$Ba_1Aa_1^{-1}B$$

$$\cdots$$

$$Ba_nAa_n^{-1}B$$

and each surface component is put into a canonical form, like the pinched sphere, by the introduction of an insion, permuting and circulating. An alternative procedure would be to subdivide each a_i with a new and regular vertex. Canonical form:

$$a_1Aa_1^{-1}b_1Bb_1^{-1}$$



$$\cdots$$

$$a_n A a_n^{-1} b_n B b_n^{-1} .$$

(5) For the multiply pinched sphere discussed earlier, we had

$$AaAbAcAdBeBe^{-1}Bd^{-1}Ac^{-1}Ab^{-1}Aa^{-1}A .$$

Introduce an incision:

$$x^{-1}CxAaAbAcAdBeBe^{-1}Bd^{-1}Ac^{-1}Ab^{-1}Aa^{-1}A ;$$

permute:

$$CxAa\cdots a^{-1}Ax^{-1}C .$$

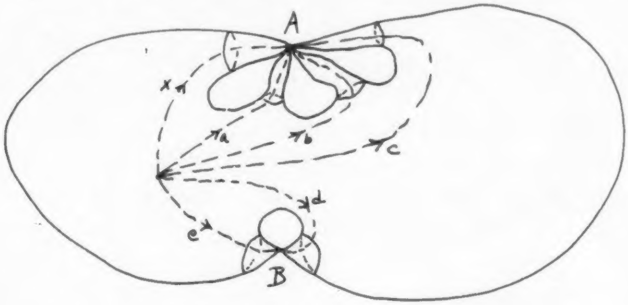
Then separate the cuffs:

$$CeBe^{-1}CdBd^{-1}CcAc^{-1}CbAb^{-1}CaAa^{-1}CxAx^{-1}C ,$$

or, omitting C:

$$eBe^{-1}dBd^{-1}cAc^{-1}bAb^{-1}aAa^{-1}xAx^{-1} .$$

This is the canonical form. Note that the small letters do not represent the same edges we started out with.



(6) For the planar complex given earlier, we have

$$Cr^{-1}Dk^{-1}C$$

$$DpEqD$$

$$AbBx^{-1}Ed^{-1}Ch^{-1}ExBeBtft^{-1}BgAc^{-1}Aa^{-1}A .$$

Now the first word has the canonical form (in the total system)

* $$yCr^{-1}Dk^{-1}Cy^{-1}$$

Similarly,

** $$zDpEqDz^{-1}$$

is the canonical form for the second word, as a member of the system. For the third word, we subdivide the edge t: $t \longrightarrow uv$, and get, after cyclic permutation:

$$Buvfv^{-1}u^{-1}BgAc^{-1}Aa^{-1}AbBx^{-1}Ed^{-1}Ch^{-1}ExBeB .$$

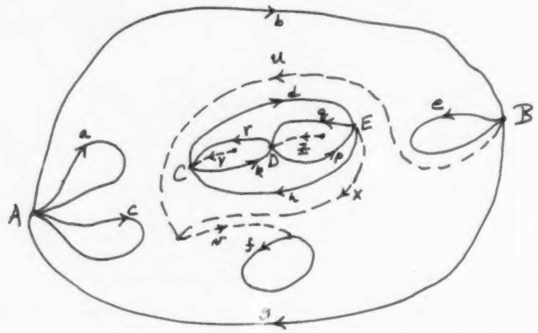
Putting Bu at the end:

$$vfv^{-1}u^{-1}BgAc^{-1}Aa^{-1}AbBx^{-1}Ed^{-1}Ch^{-1}ExBeBu \, .$$

Then, using the second circulation rule, get:

*** $$vfv^{-1}x^{-1}Ed^{-1}Ch^{-1}Exu^{-1}BgAc^{-1}Aa^{-1}AbBeBu \, .$$

Then *, **, *** constitute the canonical form.



(7) Given the surface (from an earlier example)

$$aAa^{-1}$$

$$aAbAb^{-1}Aa^{-1}$$

we "surround" the first word with an incision:

* $$xaAa^{-1}x^{-1}$$

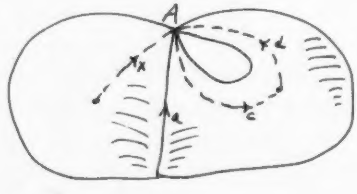
and split b, $b \longrightarrow cd$ in the second:

$$aAcdAd^{-1}c^{-1}Aa^{-1} \, ,$$

whence, by cyclic permutation:

** $$dAd^{-1}c^{-1}Aa^{-1}aAc \, .$$

Then * and ** are the canonical form.



(8) Consider the surface

$$xAx^{-1} \, , \quad Bycy^{-1}B \, , \quad BbB \, ,$$

$$aazee^{-1}z^{-1} \, ,$$

$$mkm^{-1}k^{-1}uAdAu^{-1}vee^{-1}v^{-1}wBw^{-1} \, .$$

If we split y: $y \longrightarrow rs$, we get

$$Brscs^{-1}r^{-1}B \quad \text{or} \quad scs^{-1}r^{-1}Br\ .$$

And in BbB we make an incision:

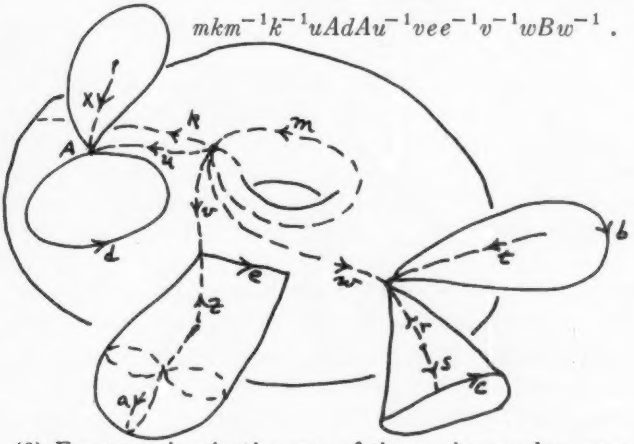
$$tBbBt^{-1}\ .$$

Then the canonical form is

$$xAx^{-1}\ , \quad scs^{-1}rBr^{-1}\ , \quad tBbBt^{-1}\ ,$$

$$aazee^{-1}z^{-1}\ ,$$

$$mkm^{-1}k^{-1}uAdAu^{-1}vee^{-1}v^{-1}wBw^{-1}\ .$$



(9) For exercise in the use of the various rules, one can take a word, phrase, sentence, paragraph of the language and, regarding it as representing a surface, put the surface in canonical form. For example, consider the word *mathematics*. Collect the crosscap $m \cdots m$:

$$mme^{-1}h^{-1}t^{-1}a^{-1}atics\ ;$$

cancel $a^{-1}a$ and $t^{-1}t$: $mme^{-1}h^{-1}ics$, and compose: $e^{-1}h^{-1}ics \longrightarrow x$, getting: mmx. Introduce an incision: $d^{-1}dmmx$, and permute: $dmmxd^{-1}$, and circulate: $mmdxd^{-1}$. This is the canonical form for a möbius band. The words *geometry*, *proof*, *algebra*, *vertex* all represent möbius bands. The word *admissible* represents a Klein bottle with a boundary. The sentence *opportunity is here* represents a sphere with four crosscaps and a boundary. The word *occurrence* has the canonical form $rreexcac^{-1}bcx^{-1}$. The sentence *jetzt habe ich ein katzenjammer* has the canonical form

$$mmz^{-1}t^{-1}e^{-1}aeqatzyece^{-1}b^{-1}a^{-1}e^{-1}t^{-1}y^{-1}\ .$$

General Finite 2-Complexes

If a 2-complex is not connected, we may, of course, study it component-wise; if some component is a linear graph, then the matter of canonical form has already been settled in the first part of this study. The matter of isolated points is not a problem. Thus there is no essential restriction

in studying just connected 2-complexes, and we make this restriction henceforth.

Let us consider a general connected finite 2-complex. It differs from the case already considered in that it may have 1-cells not incident with 2-cells. The effect on a neighborhood is to introduce the possibility that a certain number of disjoint 1-cells emanate from the center of the neighborhood. There is a *1-dimensional part* of the 2-complex, therefore, which has certain *points of junction* with the *2-dimensional part*, i.e. the subcomplex consisting of the 2-cells together with all their faces. We extend the definition of node to include the nodes defined before, also to include the points of junction (and isolated points when such are present), and the vertices of degree $\neq 2$ in the 1-dimensional part.

Now we give three listings, (0), (I), and (II) for the vertices, edges, and words, respectively. Unless there are isolated vertices, (0) can be omitted since the vertices can be read off from (I) and (II) otherwise; and such is the present case, since we assume the complex connected; moreover, (I) can be restricted to the 1-dimensional part, since the other edges can be read off from (II). Then we can put (I), the 1-dimensional part now, in canonical form as for any linear graph, preserving as vertices only nodes. (This includes the points of junction!) (II) can be put in canonical form as before (using the new definition of node; note that a node A may show up in (II) in the canonical form only as a cuff, dAd^{-1}, representing a cone leaf, and may occur nowhere else in (II); but as a point of junction, A will show up in (I) in this case).

Generalized Presentation

We have already used a single capital letter to represent a sphere, and we have also noted that we can omit the capitals for inarticulate vertices. This suggests that we can extend the mode of presentation as follows: edges as well as vertices may be labelled with capitals. Thus we allow as a word: an arbitrary finite sequence of small and/or capital letters, the small letters, of course, may, as before, have exponent -1. For example: $aABCAbca^{-1}GbR$. The capital label means that the vertex or edge receiving that label is mapped into the vertex having that label in the complex.

This generalization is a generalization of the notation only, and the presentation of the complex. For we can put any system of such general words into the corresponding alternating small and capital letter notation as follows. The capitals collect into maximal blocks (with respect to cyclic order). Two blocks may be said to be equivalent if they contain a common letter, A, or if they occupy equivalent positions with respect to the small letters (in the same way that this equivalent position was defined for capital letters earlier). This yields an equivalence class of blocks. Replace each member block of an equivalence class by a common new capital, and, furthermore, put that same capital into every equivalent position with respect to the small letters, even if no block were already present.

This last operation puts the system in the alternating form, when applied to all blocks of capitals.

The reduction above of blocks of capitals to single capitals we call *simplifying* the presentation; the reintroduction of capitals we call *completion* of the system of words. Both of these concepts are useful in studying coverings.

The simplification procedure is entirely compatible with the continuous mapping specified by the symbolism, since, if an edge is mapped into a vertex, then its ends must be sent into that same vertex. The generalized symbolism merely says that it is possible to construct a 2-complex with the collapsing of edges allowed in the process. It allows one to omit all or just part of the capitals for an inarticulate vertex. The one exception is that of the sphere when represented by a single capital. This generalized symbolism points up the fact that the canonical form is not the best for every purpose, by any means. The canonical form serves for comparison and uniqueness, but there may be simpler modes of representation, as we have seen. The canonical form represents, however, except for the "leading" edges on the cuffs, a minimal cellular decomposition of a 2-complex.

It can be shown that:

(1) two finite 2-complexes are combinatorially equivalent iff they have the same canonical form except for trivial isomorphisms;

(2) two finite 2-complexes are homeomorphic iff they have the same canonical form except for trivial isomorphisms.

The proofs of these two facts can be carried out independently (thus establishing the Hauptvermutung in two dimensions). We shall not establish these assertions, but limit ourselves to drawing information from these results.

Since the canonical form is unique to within trivial isomorphism, and since combinatorial and topological equivalence mean the same in two dimensions, we can read off from the canonical form some important topological invariants of 2-complexes: (1) the Euler characteristic: $\chi = \alpha_0 - \alpha_1 + \alpha_2$, where α_i is the number of i-cells; (2) the genus, p or q, of a surface component, i.e. the number of handles or crosscaps of any orientable, respectively non-orientable, surface component; (3) the number of cuffs on any surface component; (4) the number of nodes; (5) the number of conical points; (6) the number of point, line and surface components; (7) articulation; (8) orientability or non-orientability of a surface component; (9) the number of line components which are *closed*, i.e. which are topological circles; also the number of line components which are *open*, i.e. which are topological open 1-cells; (10) the number of leaves at a conical point, or at a line singularity, or at a node; the number of fans or cornets at a node, and so on.

The "center" of any cuff is, as we have seen, a "subword" which may be permuted cyclically. These subwords we call *singular cycles*. They are

of three mutually exclusive types:

(1) a single vertex – called a *trivial* cycle, from $d_i P d_i^{-1}$, represents a cone leaf at a node or conical point; (2) a *homogeneous* cycle – it is of the form $cc\cdots c$ where c is a closed line component; the cuff appears thus: $d_i cc\cdots cd_i^{-1}$; (3) a *noded* cycle – it is of the form $dP_1 r_1 P_2 r_2 \cdots P_k r_k P_1 d^{-1}$ where all the P_i are nodes.

A closed 2-manifold is a connected finite 2-complex every point of which is regular. It must, therefore, have exactly one of the three distinct canonical forms

$$\propto = \begin{cases} A \\ a_1 a_1 \cdots a_q a_q \\ a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_p b_p a_p^{-1} b_p^{-1} . \end{cases}$$

A bounded 2-manifold is a connected finite 2-complex which is regular except for a finite set of closed line components each having a book neighborhood of one leaf. It follows that the canonical form must be

$$\propto d_1 c_1 d_1^{-1} \cdots d_r c_r d_r^{-1} ,$$

where $\propto$ is as above. It is clear now that a 2-complex may be formed from bounded 2-manifolds (consider the canonical form): the manifolds become the surface components, their boundaries become the singular cycles in the singular graph; of course, the mapping takes place in such a way that the boundaries remain singular in the mapping process, and it may be that an entire boundary is contracted to a point. A 2-dimensional pseudomanifold is a connected finite 2-complex wherein every 1-cell is incident with precisely one or two 2-cells; it is *closed* if every 1-cell is incident with exactly two 2-cells; otherwise, it is *bounded*; thus it is bounded if some 1-cell is incident with just one 2-cell. Clearly, a manifold is a pseudomanifold. A closed pseudomanifold is either a closed manifold or all its singularities are isolated, and in the latter case the canonical form is a system of words on whose cuffs are all trivial cycles. The general form of the canonical form of a surface component in a bounded 2-pseudomanifold has certain trivial cycles, and in addition certain noded cycles,

$$dP_1 e_1 P_2 e_2 \cdots P_n e_n P_1 d^{-1} ,$$

where the line e_i occurs but once in the system of words, and possibly some homogeneous cycles, ded^{-1}, where e occurs but once in the system of words. Thus the singular cycles are a collection of closed unicursal paths, edgewise disjoint, (but not necessarily vertex-wise disjoint) covering the singular graph of the pseudomanifold, and consequently,

Theorem: *The singular graph of a pseudomanifold is an Euler graph.*

The formula $\chi = 2 - 2p - q - r$ for the characteristic of a bounded or closed 2-manifold can be generalized easily to an arbitrary finite 2-complex

as follows. Let s denote the number of surface components, n the number of nodes, h the number of closed line components, c the number of conical points, t the number of open line components, q the sum of the genuses of the non-orientable surface components, p the sum of the genuses of the orientable surface components. If we refer to the canonical form, we see that there are in it $\alpha_0 = s+c+n+h$ vertices, $\alpha_1 = 2p+q+r+h+t$ edges, and $\alpha_2 = s$ polygons, and hence, since $\chi = \alpha_0 - \alpha_1 + \alpha_2$, we get

$$\boxed{\chi = 2s+c+n-2p-q-r-t}$$

The formula applies to 0- and 1-complexes as well, for then all the above are 0 except for the number n, which is just the number of vertices of degree $\neq 2$ (included are isolated vertices), and t, so

$$\begin{cases} \chi = n-t, & \text{for 1-complexes} \\ \chi = n, & \text{for 0-complexes}. \end{cases}$$

Theorem. *If each boundary of a bounded 2-manifold is mapped homeomorphically on itself, the resulting self-homeomorphism of the total boundary can be extended to the entire 2-manifold*

(1) *in the non-orientable case, always;*

(2) *in the orientable case, if and only if the self-homeomorphisms of the particular boundaries are all orientation-preserving or all orientation-reversing.*

Proof. This follows in case (1) immediately from the trivial isomorphism (6) of the canonical form, and in case (2) from the fact that the trivial isomorphism (5) but *not* (6) is available. q.e.d.

Theorem. *If any permutation of the boundaries of a bounded 2-manifold is given, there is a self-homeomorphism of the 2-manifold which will accomplish the permutation.*

Proof. This follows from the trivial isomorphism (2). q.e.d.

Note that these two preceding theorems give us a precise description of the necessary and sufficient conditions that a self-homeomorphism of the boundary of a 2-manifold be extendable to the manifold. Moreover, the method of proof shows the topological significance of the trivial isomorphisms, and suggests also an extension of these results to a general finite 2-complex and the extension of homeomorphisms from the singular graph to the entire complex. The significance of the trivial isomorphism (3) is merely that the "approach" to a singular cycle may be made at any vertex of the cycle.

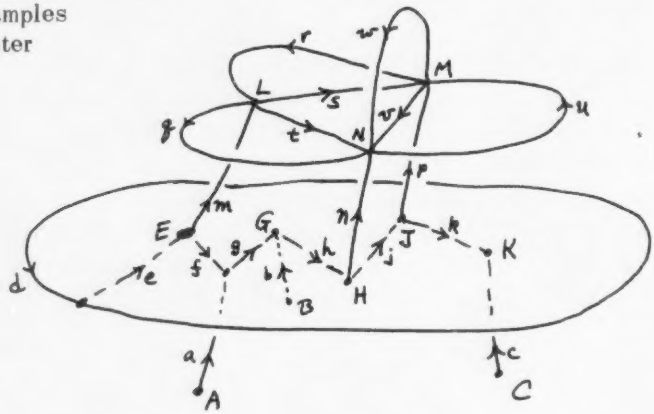
One should note that the designation "orientable" or "non-orientable" as applied to a surface component refers only to the corresponding property of the 2-manifold from which the surface component derives. Indeed, a surface component may be orientable and yet contain a möbius band. For example, consider the word $aba^{-1}b^{-1}dccd^{-1}ecce^{-1}$. This is in canonical form, and, as may always be done in the canonical form of a surface component, *one* of the leading edges can be cancelled, say e here: by circulation

past $d\cdots d^{-1}$ and the handle, bring e beside e^{-1} and cancel $e^{-1}e$, getting $aba^{-1}b^{-1}dccd^{-1}cc$. Now cut the surface in two thus: $aba^{-1}b^{-1}dccd^{-1}x^{-1}$ and xcc. The latter is a möbius band.

We have remarked before that the canonical form is a minimal cellular partition of a 2-complex, except for the leading edges, and we can see now that, in fact, a minimal cellular partition of a 2-complex is obtained by cancelling precisely one of the leading edges on each surface component.

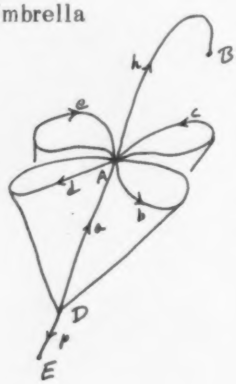
Examples

Caster



I				II
AaF	BbG	CcK		$deDfFgGhHjJkKk^{-1}Jj^{-1}Hh^{-1}Gg^{-1}Ff^{-1}Ee^{-1}$
EmL	HnN	JpM		I is in canonical form. The canonical form for II is
LqN	LtN	MrL		
LsM	MvN	MwN	NuM	$xdx^{-1}eEe^{-1}fFf^{-1}gGg^{-1}hHh^{-1}jJj^{-1}kKk^{-1}$

Umbrella



I	II
AhB DpE	$DaAdAa^{-1}D$ $DaAeAa^{-1}D$
	$DaAcAa^{-1}D$ $DaAbAa^{-1}D$

The canonical form can be obtained by surrounding each word in II with an incision.

Trinity College
Hartford 6, Conn.

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to *Joseph Seidlin, Alfred University, Alfred, New York.*

CONIC SECTIONS IN THE ELLIPTIC PLANE

Dwight B. Goodner

1. *Introduction.* Most students find the development of elliptic plane geometry and elliptic trigonometry a stimulating experience. This experience can be extended easily and naturally to include the development of an elliptic analytic geometry in a manner which parallels the development of the usual course in Euclidean analytic geometry. Unfortunately few texts attempt such a development and the usual introductory course in elliptic geometry does not make available to the student any of the powerful tools of elliptic analytics. It is our opinion that knowledge of the methods and techniques of elliptic analytic geometry will be of interest and value to the student.

The purpose of this paper is to develop and study the equations of conic sections in the elliptic plane. Our development will, in general, follow the pattern of the traditional course in Euclidean plane analytic geometry. Because of the difficulty of representation, we will explore the elliptic plane by analytics only; all figures used are for the convenience of the reader and are meant only to aid him in understanding the relative positions of the various elements.

For our study we will use the Weierstrassian point and line coordinates. Then the equation of a straight line will be of the first degree. We define a conic section or, more simply, a conic as the locus of a homogeneous equation of the second degree; that is, a conic section is a curve of the second degree.

In Euclidean geometry the conic sections are classified with reference to the manner in which they cut the line at infinity. In non-Euclidean geometry conic sections are classified in an analogous manner; that is, they are classified with reference to their intersections with the absolute. Since the absolute in elliptic geometry is imaginary, a conic section with a real trace can cut it only in imaginary points. Therefore, we can have, strictly speaking, only degenerate forms, proper circles, and ellipses. However, for convenience, an equation developed from a characteristic property of

the Euclidean parabola or hyperbola will be designated as the equation of the elliptic "parabola" or "hyperbola".

2. *Coordinates.* The coordinates which we will use are the Weierstrassian point and line coordinates. The point coordinates [2, p. 127]

$$x = k\sin u/k = k\sin r/k\cos\theta$$

$$y = k\sin v/k = k\sin r/k\sin\theta$$

$$z = \cos r/k$$

are connected by the relation

$$x^2 + y^2 + k^2 z^2 = k^2 \tag{1}$$

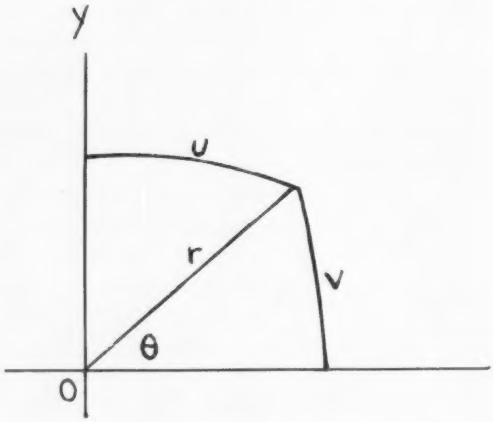


Figure 1

and the line coordinates [2, p. 129]

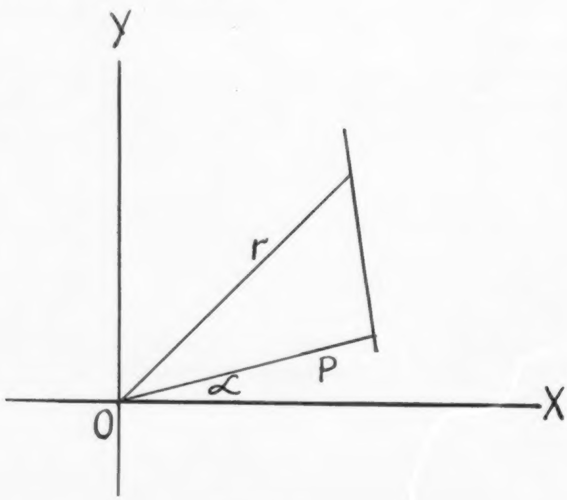


Figure 2

$$\xi = \cos \alpha \cos p/k$$
$$\eta = \sin \alpha \cos p/k$$
$$\zeta = -k \sin p/k$$

are connected by the relation

$$k^2\xi^2 + k^2\eta^2 + \zeta^2 = k^2 .$$

This choice of coordinates enables us to write the equation of a straight line in the form

$$\xi x + \eta y + \zeta z = 0 \quad [2, \text{p. } 129] .$$

Since any equation in these coordinates may be made homogeneous by using relations (1) and (2), we may, in general, use ratios of the coordinates. However, unless the contrary is indicated, we will use exact coordinates in this paper.

3. *Distance Formulas*. The distance d between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by [2, p. 129]

$$\cos\frac{d}{k} = \frac{x_1x_2 + y_1y_2 + k^2 z_1 z_2}{k^2} . \tag{3}$$

A useful derivation of this formula is

$$k^2\sin^2\frac{d}{k} = \frac{1}{k^2}\begin{vmatrix} x_1 y_1 \\ x_2 y_2 \end{vmatrix}^2 + \begin{vmatrix} x_1 z_1 \\ x_2 z_2 \end{vmatrix}^2 + \begin{vmatrix} y_1 z_1 \\ y_2 z_2 \end{vmatrix}^2 . \tag{3'}$$

The distance d of the point (x, y, z) from the line $\xi x + \eta y + \zeta z = 0$ is given by [2, p. 132]

$$\sin\frac{d}{k} = \frac{\xi x + \eta y + \zeta z}{k} . \tag{4}$$

4. *Transformations of Coordinates*. In this section we develop formulas for the translation and rotation of axes [1, p. 164-170]. The reader can easily verify that distance and angle are invariant under these transformations.

Let the $X'Y'$ coordin[illegible] system be determined by the rectangular axes $0'X'$ and $0'Y'$ where $0'$ has the coordinates $(0, y_1, z_1)$, $0'X'$ has the equation $z_1y - y_1z = 0$ and $0'Y'$ has the equation $x = 0$ when referred to the XY coordinate system. Let the point P have coordinates (x, y, z) when referred to the XY coordinate system and (x', y', z') when referred to the $X'Y'$ coordinate system. Then by the law of sines [2, p. 120; 3, p. 196] we obtain

$$x = k \sin r/k \cos\theta = k \sin r'/k \cos\theta' = x'$$

and by the law of cosines we obtain

$$z = \cos r/k = \cos d/k \cos r'/k + \sin d/k \sin r'/k \cos(\theta' + \pi/2)$$

$$= \frac{k^2 z_1 z' - y_1 y'}{k^2}$$

and

$$y = k \sin r/k \sin\theta = k \sin r/k \left(\frac{\cos r'/k - \cos d/k \cos r/k}{\sin d/k \sin r/k}\right)$$

$$= k\left(\frac{z' - z_1 z}{y_1/k}\right) = y_1 z' + z_1 y'.$$

Hence the XY coordinates and the $X'Y'$ coordinates are connected by the

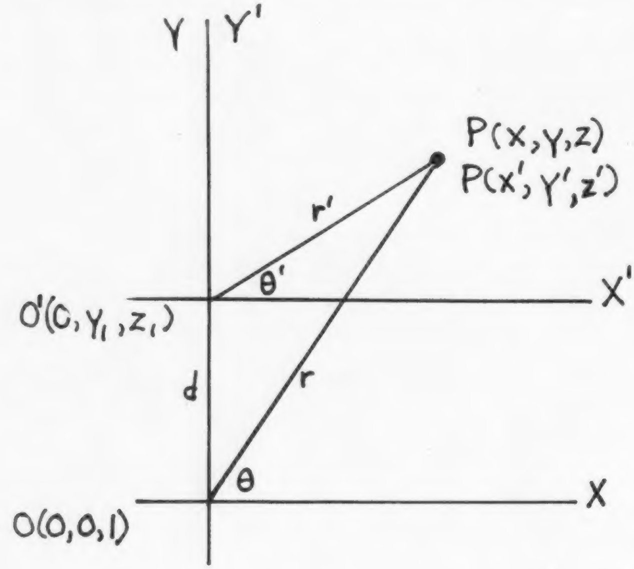


Figure 3.

relations

$$x = x'$$

(5) $$y = y_1 z' + z_1 y'$$

$$z = \frac{k^2 z_1 z' - y_1 y'}{k^2}.$$

It is easy to see that

$$x' = x$$

(5′) $$y' = -y_1 z + z_1 y$$

$$z' = \frac{k^2 z_1 z + y_1 y}{k^2}.$$

Let the $X''Y''$ coordinate system be determined by the rectangular axes $0''X''$ and $0''Y''$ where $0''$ has the coordinates $(x_2, 0, z_2)$, $0''Y''$ has the equation $z_2 x - x_2 z = 0$ and $0''X''$ has the equation $y'' = 0$ when

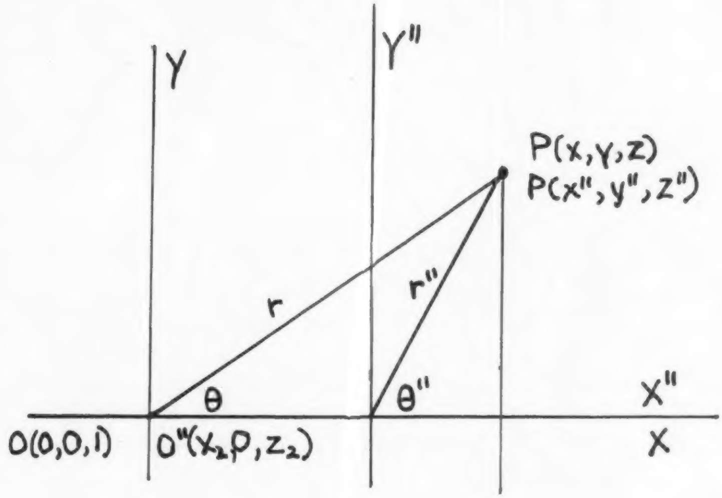


Figure 4.

referred to the XY coordinate system. If the point P has the coordinates (x, y, z) when referred to the XY coordinate system and (x'', y'', z'') when referred to the $X''Y''$ coordinate system, then

$$x = x_2 z'' + z_2 x''$$

(6) $$y = y''$$

$$z = \frac{k^2 z_2 z'' - x_2 x''}{k^2}$$

and

$$x'' = -x_2 z + x z_2$$

(6′) $$y'' = y$$

$$z'' = \frac{k^2 z_2 z + x_2 x}{k^2}.$$

Transformations (5) and (6) may be combined to give the equations of a general translation. Let the $X'''Y'''$ coordinate system be determined by

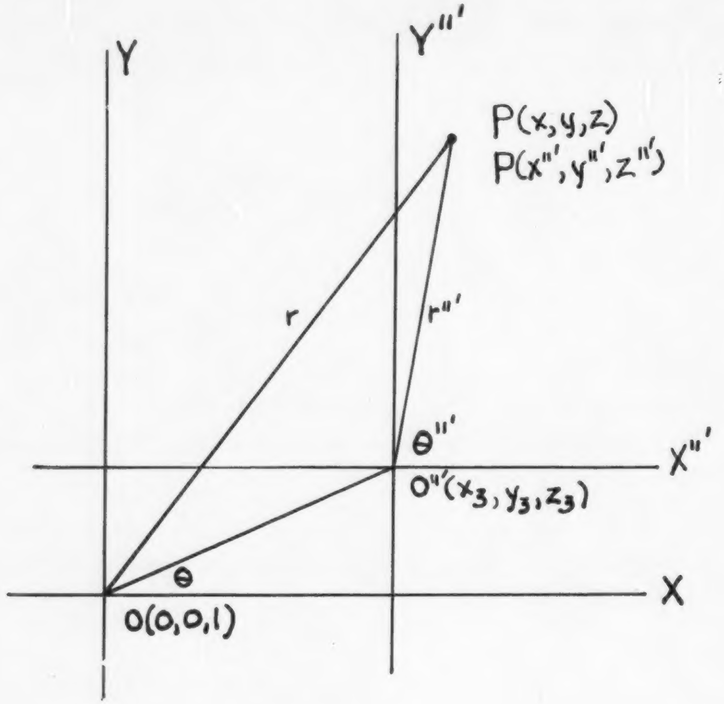


Figure 5.

the rectangular axes $0'''X'''$ and $0'''Y'''$ where $0'''$ has the coordinates $(x_3, y_3, z_3) \neq (0, k, 0)$, $0'''X'''$ has the equation

$$x_3y_3x-(x_3^2+k^2z_3^2)y+k^2y_3z_3z = 0$$

and $0'''Y'''$ has the equation

$$z_3x-x_3z = 0$$

when referred to the XY coordinate system. From the transformations above we obtain

$$x = x_3z'''-\frac{x_3y_3y'''}{k\sqrt{x_3^2+k^2z_3^2}}+\frac{kz_3x'''}{\sqrt{x_3^2+k^2z_3^2}}$$

(7) $$y = y_3z'''+\frac{\sqrt{x_3^2+k^2z_3^2}}{k}y'''$$

$$z = z_3 z''' - \frac{y_3 z_3 y'''}{k\sqrt{x_3^2 + k^2 z_3^2}} - \frac{x_3 x'''}{k\sqrt{x_3^2 + k^2 z_3^2}}.$$

The inverse transformation is

$$x''' = \frac{k}{\sqrt{x_3^2 + k^2 z_3^2}}(z_3 x - x_3 z)$$

(7′)
$$y''' = \frac{\sqrt{x_3^2 + k^2 z_3^2}}{k} y - \frac{y_3(k^2 z_3 z + x_3 x)}{k\sqrt{x_3^2 + k^2 z_3^2}}$$

$$z''' = \frac{k^2 z_3 z + y_3 y + x_3 x}{k^2}.$$

We observe that if $(x_3, y_3, z_3) \not\equiv (k, 0, 0)$, the lines with equations $z_3 y - y_3 z = 0$ and $(y_3^2 + k^2 z_3^2)x - x_3 y_3 y - k^2 x_3 z_3 z = 0$ could have been taken for $0'''X'''$ and $0'''Y'''$ respectively. We leave to the reader the development of the equations of the transformation and its inverse.

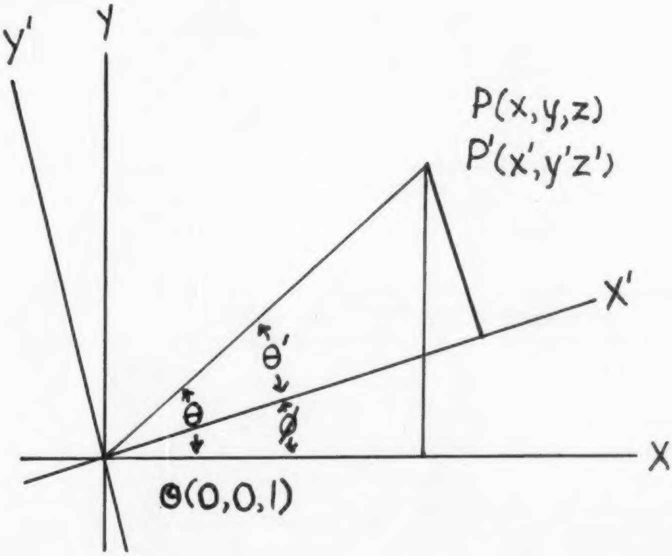


Figure 6.

The transformation for the rotation of axes is similar to the transformation used for the rotation of axes in the Euclidean plane. Let the $X'Y'$ coordinate system be determined by the rectangular axes $0X'$ and $0Y$ and let ϕ be the angle between $0X$ and $0X'$. Since $\theta = \theta' + \phi$ and $r = r'$, we

have

$$x = k \sin r/k \cos\theta = k \sin r/k (\cos\theta' \cos\phi - \sin\theta' \sin\phi)$$
$$y = k \sin r/k \sin\theta = k \sin r/k (\sin\theta' \cos\phi + \cos\theta' \sin\phi)$$
$$z = z' .$$

Hence

$$\begin{aligned} x &= x' \cos\phi - y' \sin\phi \\ y &= x' \sin\phi + y' \cos\phi \\ z &= z' \end{aligned} \tag{9}$$

and

$$\begin{aligned} x' &= x \cos\phi + y \sin\phi \\ y' &= y \cos\phi - x \sin\phi \\ z' &= z. \end{aligned} \tag{9'}$$

5. *The Elliptic Circle.* A circle is the set of points equidistant from a fixed point. Let $P_0(x_0, y_0, z_0)$ be the fixed point (center) and r the fixed distance (radius). If $P(x, y, z)$ is a generic point of the circle, we obtain from (3) the equation

$$k^2 \cos r/k = x_0 x + y_0 y + k^2 z_0 z \ . \tag{10}$$

If $r = \pi k/2$, the equation becomes

$$x_0 x + y_0 y + k^2 z_0 z = 0$$

which is the equation of a straight line. The line is the absolute pole [2, p. 89] of the center of the circle. From equation (10) we see that an equation of the form

$$ax + by + cz + d = 0$$

is the equation of a proper circle provided $0 < d^2 \leqq k^2a^2 + k^2b^2 + c^2$. The center and the radius may be determined from the relations

$$x_0 = \frac{k^2 a}{\sqrt{k^2a^2 + k^2b^2 + c^2}} ,$$

$$y_0 = \frac{k^2 b}{\sqrt{k^2a^2 + k^2b^2 + c^2}} , \quad z_0 = \frac{c}{\sqrt{k^2a^2 + k^2b^2 + c^2}}$$

and

$$\cos r/k = \frac{-d}{\sqrt{k^2a^2 + k^2b^2 + c^2}} .$$

Another form of the equation of the circle may be obtained from (3′). If $P_0(x_0, y_0, z_0)$, r and $P(x, y, z)$ are the same as above, then

(11) $$k^2 \sin^2 r/k = \frac{1}{k^2}\begin{vmatrix} x & y \\ x_0 & y_0 \end{vmatrix}^2 + \begin{vmatrix} x & z \\ x_0 & z_0 \end{vmatrix}^2 + \begin{vmatrix} y & z \\ y_0 & z_0 \end{vmatrix}^2 .$$

We observe that as k tends to infinity this expression tends to the limiting form

$$(x-x_0)^2+(y-y_0)^2 = r^2 .$$

6. *Standard Form of the Equation of the Elliptic Ellipse*. The Euclidean Ellipse has the important property that the sum of the distances from each point of the ellipse to two fixed points is constant. We shall call any elliptic conic section which has this property an elliptic ellipse and we shall use the property to develop the standard form of the equation of the elliptic ellipse. Let the points $F_1(n, 0, l)$ and $F_2(-n, 0, l)$ be the fixed

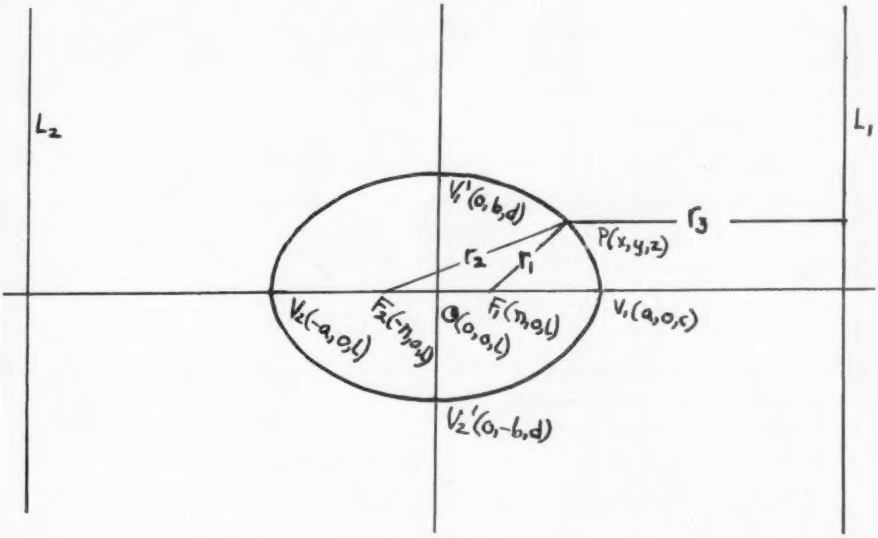


Figure 7.

points (foci), let the point $P(x, y, z)$ be a generic point of the ellipse, let r_1 be the distance F_1P, and let r_2 be the distance F_2P. Let $r_1+r_2 = 2a_1$ where a_1 is a positive constant less than or equal to $\pi k/2$, and let $a = k\sin a_1/k$ and $c = \cos a_1/k$. Then

(8) $$\cos\frac{r_1+r_2}{k} = 2c^2-1 .$$

If $a_1 = \pi k/2$, we obtain the degenerate conic $z^2 = 0$. If $a_1 < \pi k/2$, we obtain

(9) $$(a^2c^2-n^2c^2)x^2+a^2c^2y^2+(k^2a^2c^2-k^2a^2l^2)z^2 = 0 .$$

Elimination of c, l, and z by means of relation (1) gives us the standard form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{9'}$$

where

$$b^2=\frac{k^2(a^2-n^2)}{k^2-n^2}.$$

The ellipse intersects the x-axis at the points $V_1(a, 0, c)$ and $V_2(-a, 0, c)$ and intersects the y-axis at the points $V_1'(0, b, d)$ and $V_2'(0, -b, d)$. The points V_1 and V_2 are called the vertices of the ellipse, the line segment $V_2 0 V_1$ is called the major axis, and the line segment $V_2' 0 V_1'$ is called the minor axis. The intersection of the major and minor axes, $0(0, 0, 1)$, is called the internal center of the ellipse and the point $0'(k, 0, 0)$ is called the external center of the ellipse. We note that 0 and $0'$ are the midpoints of $V_2 0 V_1$ and $V_2 0' V_1$ respectively.

The line determined by the points $F_2(-n, 0, l)$ and $P(x, y, z)$ is divided by these points into two parts of length r_2 and $\pi/k - r_2$ [2, p. 201]. Let $r_3 = \pi k - r_2$. Then $r_2 = \pi k - r_3$. Substituting in (8) we obtain

$$\cos\frac{r_1-r_3}{k}=1-2c^2$$

which shows that the equation of the elliptic ellipse can be obtained using the difference of two distances.

Theorem 6.1. The difference of the distances from each point of an elliptic ellipse to two fixed points (foci) is constant.

Let r_3 be the distance from the point $P(x, y, z)$ to the line L_1 whose equation is $c^2nx-a^2lz=0$ and, as before, let r_1 be the distance from P to the point $F_1(n, 0, l)$. If we express $\sin r_1/k$ and $\sin r_3/k$ in terms of coordinates by means of formulas (3′) and (4) respectively, it is easy to show that

$$\frac{\sin r_3/k}{\sin r_1/k}=\sqrt{\frac{k^2a^2c^2}{a^4l^2+k^2c^4n^2}}\ ;$$

that is, the ratio of the sines is constant [2, p. 262]. This expression may be used to obtain equation (9). The point $F_2(-n, 0, l)$ and the line L_2 whose equation is $c^2nx+a^2lz=0$ could have been used in place of F_1 and L_1 respectively. We state these results as a theorem.

Theorem 6.2. The ratio of the sines of the distances (divided by k)

from each point of an elliptic ellipse to a fixed line (directrix) and a fixed point (focus) is constant.

We obtain the relation $c = ld$ by substituting 0, b, and d for x, y, and z respectively in (9). If n_1 is the length of $0F_1$, b_1 is the length of $0V_1'$ and w_1 is the length of $V_1'F_1$, we see from the above relation and the right triangle $F_1 0 V_1'$ that $\cos w_1/k = \cos n_1/k \cos b_1/k = \cos a_1/k$; that is, we see that $w_1 = a_1$. We state this result as a theorem.

Theorem 6.3. In an elliptic ellipse the distance from a focus to an end of the minor axis is equal to one-half the length of the major axis.

We observe that if k tends to infinity, the equation (9) of the elliptic ellipse tends to the limiting form

$$b_1^2 x^2 + a_1^2 y^2 = a_1^2 b_1^2 \ .$$

7. *Standard Form of the Equation of the Elliptic "Parabola"*. Each point of an Euclidean parabola is equally distant from a fixed point and a fixed line. We shall use this property to develop the equation of the elliptic

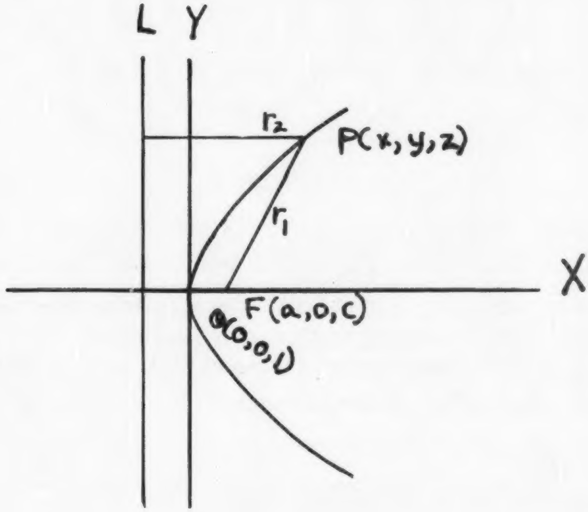


Figure 8.

"parabola". Let the point $F(a, 0, c)$, $a < \pi k/2$, be the fixed point (focus), let the line L whose equation is $cx + az = 0$ be the fixed line (directrix), let the point $P(x, y, z)$ be a generic point of the "parabola", let r_1 be the distance from P to F, and let r_2 be the distance from P to L. Then

$$\sin r_1/k = \sin r_2/k$$

and from this we obtain

$$y^2 = 4acxz$$

which we shall call the standard form of the equation of the elliptic "parabola". Elimination of c and z from the equation gives an irreducible equation of the fourth degree which does not contribute to our discussion.

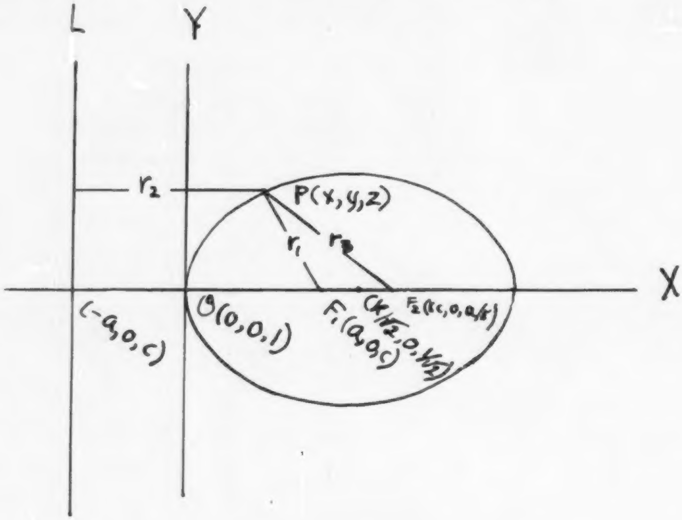


Figure 9.

The point $F_2(kc,\ 0,\ a/k)$ is the pole [2, p. 92] of L. Then if r_3 is the distance from P to F_2, $r_2 + r_3 = \pi k/2$ and $r_2 = \pi k/2 - r_3$. Hence if $r_1 = r_2$, $r_1 = \pi k/2 - r_3$ or $r_1 + r_2 = \pi k/2$. Thus we see that the elliptic "parabola" is an elliptic ellipse with major axis of length $\pi k/2$.

We observe that if k tends to infinity, the equation (10) of the elliptic "parabola" tends to the limiting form

$$y^2 = 4a_1 x \ .$$

8. *Standard Form of the Equation of the Elliptic "Hyperbola"*. The difference of the distances from each point of an Euclidean hyperbola to two fixed points (foci) is constant. We shall use this property of the Euclidean hyperbola to develop the standard form of the equation of the elliptic "hyperbola". Let the points $F_1(n,\ 0,\ l)$ and $F_2(-n,\ 0,\ l)$, $n < \pi k/2$, be the fixed points (foci), let the point $P(x,\ y,\ z)$ be a generic point of the "hyperbola", let r_1 be the distance F_1P and let r_2 be the distance F_2P. Let $|r_1 - r_2| = 2a_1$ where a_1 is a constant and let $a = k \sin a_1/k$ and $c = \cos a_1/k$. Then

$$\cos \frac{r_1 - r_2}{k} = 2c^2 - 1 \tag{11}$$

and from this we obtain

(12) $$(a^2c^2 - n^2c^2)x^2 + a^2c^2y^2 + (k^2a^2c^2 - k^2a^2l^2)z^2 = 0$$

which is the same as equation (9) of the elliptic ellipse. However, elimi-

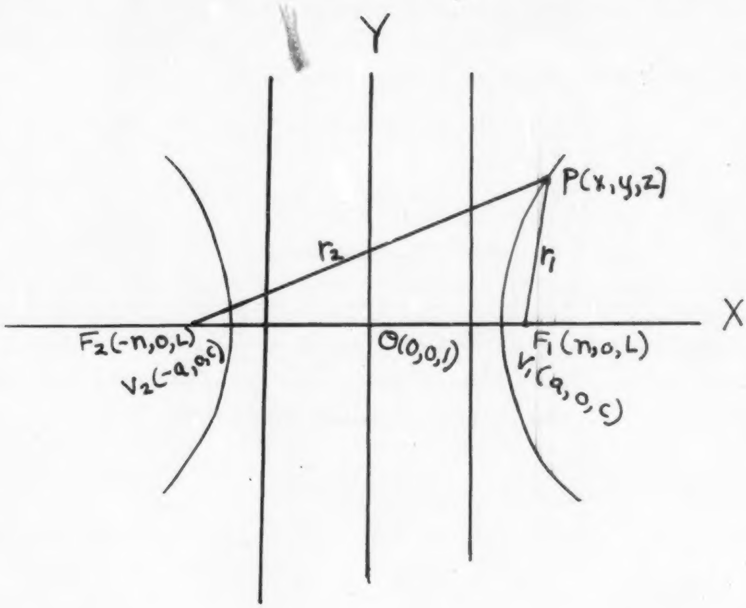


Figure 10.

nation of c, l and z gives us the standard form

(12′) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where

$$b^2 = \frac{k^2(n^2 - a^2)}{k^2 - n^2}.$$

The line determined by F_2 and P is divided by these points into two parts of lengths r_2 and $\pi k - r_2$ [2, p. 102]. Let $r_3 = \pi k - r_2$. Then $r_2 = \pi k - r_3$. Substituting these in (11) we obtain

$$\cos\frac{r_1 + r_3}{k} = 1 - 2c^2$$

which shows that the elliptic "hyperbola" is an elliptic ellipse. The elliptic "hyperbola" in standard position is an elliptic ellipse with internal center at the point $0'(k, 0, 0)$ and external center at the origin $0(0, 0, 1)$.

We observe that if k tends to infinity, the equation (12) of the elliptic hyperbola tends to the limiting form

$$b_1^2x^2 - a_1^2y^2 = a_1^2b_1^2 .$$

9. *The General Equation of the Second Degree.* The transformations developed in section 4 enable us to reduce the general homogeneous equation of the second degree

$$Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2 = 0 \tag{13}$$

to the form

$$Px'^2 + Qy'^2 = R \tag{13'}$$

where $P \cdot Q \geqq 0$. If equation (13) can be expressed as the product of linear factors with real coefficients the reduction can be made without difficulty. If equation (13) cannot be expressed as the product of linear factors with real coefficients, the reduction may be accomplished by eliminating the xy term by the rotation of axes, writing the equation in the form $(px-qz)^2 \pm (ry-sz)^2 = t$, translating the axes so that the point

$$\left(\frac{kqr}{\sqrt{q^2r^2+p^2s^2+k^2p^2r^2}}, \frac{kps}{\sqrt{q^2r^2+p^2s^2+k^2p^2r^2}}, \frac{kpr}{\sqrt{q^2r^2+p^2s^2+k^2p^2r^2}}\right)$$

is the new origin (the equation will take the form $A'x^2 + B'xy + C'y^2 = K$), and then eliminating the xy term. If the coefficients of the x^2 and y^2 terms have different signs, translating the origin to the point $(k, 0, 0)$ will give an equation of the desired form.

If equation (13) has a real nondegenerate locus, the locus is a proper circle or an ellipse. Hence study of the nondegenerate conic sections of the elliptic plane reduces to study of the elliptic circle and the elliptic ellipse.

REFERENCES

1. H. Busemann, "Non-Euclidean Geometry," *The Tree of Mathematics*, ed. by G. James, Pacoima, The Digest Press, 1957, pp. 156-172.

2. D. M. Y. Sommerville, *The Elements of Non-Euclidean Geometry*, New York, Dover Publications, Inc., 1958.

3. H. E. Wolfe, *Introduction to Non-Euclidean Geometry*, New York, The Dryden Press, 1948.

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MISCELLANEOUS NOTES

Edited by

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Articles intended for this department should be sent to *Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Ind.*

TREE OF COMPOSITIONS

Irving J. Gabelman

A collection of integers, whose sum is n, is defined as a partition of the number n. If the collections are ordered, they become the compositions of the number n. Partitions and compositions are part of the subject matter of Combinatorial Analysis and rather extensive treatments of their properties may be found in the literature[1,2]. The partitions and compositions

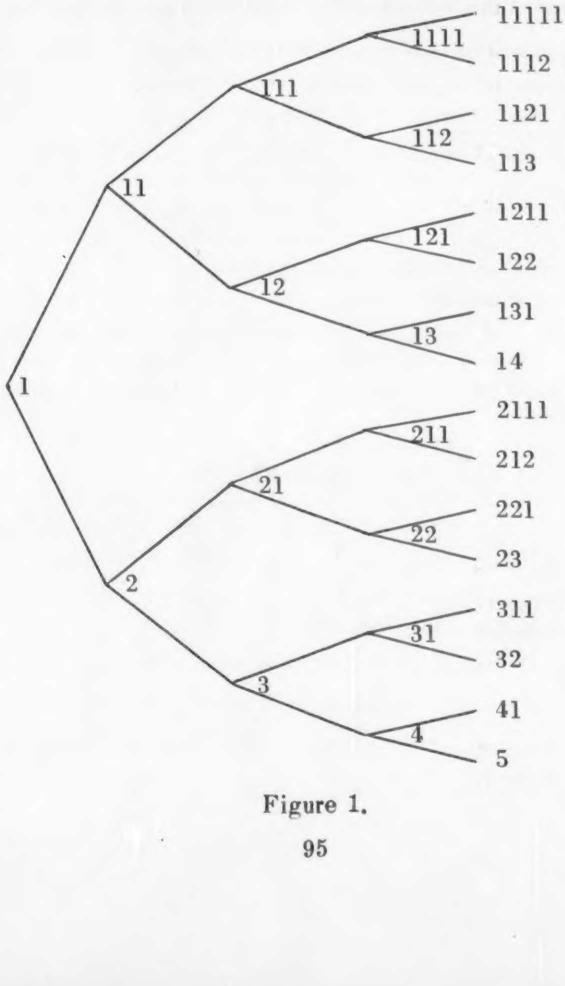

Figure 1.

of the integers 1, 2, 3, 4, 5 are given in Table 1.

Number	Partitions	Compositions
1	1	1
2	2, 11	2, 11
3	3, 21, 111	3, 21, 12, 111
4	4, 31, 22, 211, 1111	4, 31, 22, 211, 13, 121, 112, 1111
5	5, 41, 32, 311, 221, 2111, 11111	5, 41, 32, 311, 23, 221, 212, 2111, 14, 131, 122, 1211, 113, 1121, 1112, 11111

Table I

Here the compositions of the integer n with m parts is given by $C_{m,n} = \binom{n-1}{m-1}$ and the total number of compositions is $\sum_{m=1}^{n} C_{m,n} = 2^{n-1}$.

A systematic and interesting method of enumerating the compositions of the number n is by constructing a tree of compositions. The tree is so named because the vertices of level L_n of the tree are labelled with the compositions of the number n. A simple way of labelling these vertices is the following. Designate a general vertex at level L_n as $\{C_1, C_2, \cdots, C_{n-1}, C_n\}$. The vertex at L_{n+1} (the level to the right of L_n) immediately connected and below is labelled $C_1, C_2, \cdots, C_n + 1$ while the vertex above at level L_{n+1} is labelled $C_1, C_2, \cdots, C_{n-1}, C_n, 1$. The covering vertex is labelled 1. The tree developed in this manner to L_5 is shown in Figure 1.

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[1] *Combinatorial Analysis* – P. A. MacMahon

[2] *An Introduction to Combinatorial Analysis* – John Riordan

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HIGHER ORDER APPROXIMATIONS TO SOLUTIONS OF TRANSCENDENTAL SYSTEMS

C. E. Maley

A recent paper by Wolfe[1] prompts the discussion of higher order approximations of the zeros of functions when the derivatives are not accessible.

A convenient notation[2] will be used:

$$P(1,\ x,\ x^2)_i \equiv \begin{bmatrix} 1 & x_{i-2} & x_{i-2}^2 \\ 1 & x_{i-1} & x_{i-1}^2 \\ 1 & x_i & x_i^2 \end{bmatrix},$$

$\phi(1,\ x,\ x^2)_i$ is the corresponding determinant and, for example,

$$\phi(1,\ x,\ x^2)_i^{i-1} \equiv \begin{vmatrix} 1 & x_{i-2} & x_{i-2}^2 \\ 1 & x & x^2 \\ 1 & x_i & x_i^2 \end{vmatrix}.$$

Finally,

$$P(X,\ X^2)_i \equiv \begin{bmatrix} x_{i-2}-x_i & (x_{i-2}-x_i)^2 \\ x_{i-1}-x_i & (x_{i-1}-x_i)^2 \end{bmatrix}.$$

If $(x_{i-2},\ f_{i-2})$, $(x_{i-1},\ f_{i-1})$, $(x_i,\ f_i)$ are three points in a neighborhood of x such that

$$f(x) = 0\ , \tag{1}$$

one may elect to pass an approximating parabola through the points of either type,

$$f(x) \doteqdot a + bx + cx^2$$

or

$$x(f) \doteqdot m + nf + pf^2\ .$$

Here $\doteqdot$ indicates quadratic approximation.

This situation may be summarized as

$$[1\ f\ f^2]P^{-1}(1,\ f,\ f^2)_i \doteqdot [1\ x\ x^2]P^{-1}(1,\ x,\ x^2)_i\ . \tag{2}$$

Cramer's rule gives the determinantal form

$$[\phi(1, f, f^2)_i^{i-2}\ \phi(1, f, f^2)_i^{i-2}\phi(1, f, f^2)_i^{i-1}\phi(1, f, f^2)_i^{i}]/\phi(1, f, f^2)_i \doteq$$

$$[\phi(1, x, x^2)_i^{i-2}\phi(1, x, x^2)_i^{i-1}\phi(1, x, x^2)_i^{i}]/\phi(1, x, x^2)_i\,.$$

Thus transformation (2) is one-one only at the three base points.

Postmultiplying both sides of (2) by $P(1, x, x^2)_i$ and substituting $x = x_{i+1}$, $f = 0$, one gets the direct quadratic interpolation formula for a zero of (1):

$$x_{i+1} \doteq [1\ 0\ 0]P^{-1}(1, f, f^2)_i P(x)_i\,.$$

Postmultiplying (2) by $P(1, f, f^2)_i$ and making the same substitutions, one gets the inverse quadratic formula:

$$[1\ x_{i+1}\ x_{i+1}^2]P^{-1}(1, x, x^2)_i P(f)_i \doteq 0\,. \tag{3}$$

Inverse quadratic interpolation for roots has been considered by Müller[3], and Dandelin[4].

Matrix operations of the type $AB^{-1}C$, where B is nonsingular, may be conveniently done by using the "biaugmented matrix" $\begin{matrix} A & \\ B & C\end{matrix}$. B may then be reduced to the identity matrix by column-equivalent operations on A and B, and/or row operations on B and C.

Or, by Cramer's rule,

$$P^{-1}(1, x, x^2)_i P(1, f, f^2)_i \equiv \phi^{-1}(1, x, x^2)_i \begin{vmatrix} 1 & \phi(f, x, x^2) & \phi(f^2, x, x^2) \\ 0 & \phi(1, f, x^2) & \phi(1, f^2, x^2) \\ 0 & \phi(1, x, f) & \phi(1, x, f^2) \end{vmatrix}_i ,$$

so that these methods become

$$x_{i+1} \doteq \frac{\phi(x, f, f^2)_i}{\phi(1, f, f^2)_i} \tag{4}$$

and

$$\phi(f, x, x^2)_i + x_{i+1}\phi(1, f, x^2)_i + x_{i+1}^2\phi(1, x, f)_i \doteq 0\,. \tag{5}$$

These are extensions, to higher direct and inverse orders, of the secant method (method of "false position"),

$$x_{i+1} \doteq \frac{\phi(x, f)_i}{\phi(1, f)_i}, \tag{6}$$

which is self inverse.

Extensions to higher dimensions are possible. Thus, for the system

$$f(x,\ y) = 0$$
$$g(x,\ y) = 0\ ,$$

the direct quadratic method is got from

$$[x - x_i\ \ y - y_i] \doteq [f - f_i\ \ g - g_i\ \ (f - f_i)^2\ \ 2(f - f_i)(g - g_i)\ \ (g - g_i)^2]$$
$$P^{-1}(F,\ G,\ F^2,\ 2FG,\ G^2)_i P(X,\ Y)_i\ .$$

Then

$$[x\ y]_{i+1} \doteq [x\ y]_i + [-f\ \ -g\ \ f^2\ \ 2fg\ \ g^2]_i P^{-1}(F,\ G,\ F^2,\ 2FG,\ G^2)_i P(X,\ Y)_i\ .$$

The linear method, in determinantal form, is

$$x_{i+1} \doteq \frac{\phi(x,\ f,\ g)_i}{\phi(1,\ f,\ g)_i},\quad y_{i+1} \doteq \frac{\phi(y,\ f,\ g)_i}{\phi(1,\ f,\ g)_i}\ .$$

As a numerical illustration, consider the solution of

$$f(x) \equiv 2\sin x - x = 0$$

for the root closest to $x = 2$. Methods (6) and (4) will be used:

method:	(6)		(4)	
i	x	f	x	f
-2			2.2	-0.58300716
-1	1.8	0.14769527	1.8	0.14769527
0	2	-0.18149514	2	-0.18140514
1	1.8897569	0.00936681	1.8937798	0.00280565
2	1.8951698	0.00053141	1.8955968	-0.00002044
3	1.8954953	-0.00000174	1.8954943	0.00000001
4	1.8954942	0.00000006		
5	1.8954943	0.00000001		

Method (4) is of particular usefulness when a good guess is not possible.

REFERENCES

1. Wolfe, J. M. A Determinant Formula for Higher Order Approximations of Roots. Mathematics Magazine, March-April 1958.
2. Maley, C. E. A Simplified Numerical Analysis. Journal of the Royal Aeronautical Society, January 1959.
3. Müller, D. E. A Method for Solving Algebraic Equations Using an Automatic Computer. Mathematical Tables and Other Aids to Computation, October 1956.
4. Dandelin, G. P. Recherches sur la resolution des equations numeriques. Nouv. Mem. Acad. Roy. Sci. Belles-Lettres de Bruxelles, Vol. 3, 1826.

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ON THE DIGITAL ROOTS OF PERFECT NUMBERS

Maxey Brooke

The digital root of a number (a term apparently introduced by H. E. Dudney) is the ultimate sum of its digits. For example, the digital root of 41 is

$$4+1=5\ .$$

The digital root of 583 is

$$5+8+3=16\ ;\quad 1+6=7\ .$$

With the exception of "casting out nines", digital roots have been neglected in the theory of numbers.

Digital roots obey the following rules:

1. The digital root of the sum of two numbers equals the sum of the digital roots of the numbers.

2. The digital root of the product of two numbers equals the product of the digital roots of the numbers.

A most interesting property of digital roots is their ability to form repeating series. Take for example, the squares of the first nine numbers;

$$1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81.$$

The digital roots of the squares are

$$1,\ 4,\ 9,\ 7,\ 7,\ 9,\ 4,\ 1,\ 9.$$

The squares of the next nine numbers are

$$100,\ 121,\ 144,\ 169,\ 196,\ 225,\ 256,\ 289,\ 324.$$

The sequence of the digital roots is repeated;

$$1,\ 4,\ 9,\ 7,\ 7,\ 9,\ 4,\ 1,\ 9.$$

And so on for the next nine squares, etc.

It was first noted by inspection that all known perfect numbers greater than 6 have the digital root of 1.

The formula for a perfect number must be

$$2^{n-1}(2^n-1)$$

where 2^n-1 is prime.

Now, the digital root of the term 2^n forms the sequence

$$2,\ 4,\ 8,\ 7,\ 5,\ 1$$

and the digital roots of 2^n-1 will form the sequence

$$1,\ 3,\ 7,\ 6,\ 4,\ 9.$$

(Continued on page 124)

CURRENT PAPERS AND BOOKS

Edited by H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to *H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.*

REPRESENTATION OF n^r BY n^p CONSECUTIVE GNOMONS

Russell V. Parker

The article "On Unequal Partitions of Integers" by Iwao Sugai (this magazine, Vol. 33, No. 3, pp. 129-138) discusses a very interesting problem. The further remarks contained in the present article deal with consecutive gnomons.

By its nature a gnomon is an odd number. The problem therefore resolves itself into one of representing the number n^r (n and r positive integers) as the sum of consecutive odd numbers. It is well known that the series of odd numbers of the form $2n-1$ generates the sequence of square numbers, i.e., n^2. We have:

	1	+	3	+	5	+	7	+	9	+	11	+	13	+	15	+ ...
Sums:	1		4		9		16		25		36		49		64	...

which can be expressed as:

$$\sum_{t=1}^{n} (2t-1) = 2\left(\frac{n^2}{2} + \frac{n}{2}\right) - n = n^2 .$$

In the case of n^2, the sets of consecutive odd numbers "overlap", since each set begins with 1. This need not be so for higher powers of n. We need the set which represents n^r to consist of precisely n odd numbers, which are to be consecutive. Suppose that the set in question runs from the $(a+1)$ st odd number to the b th odd number. Then:

$$b^2 - a^2 = n^r = n \cdot n^{r-1} = (b+a)(b-a)$$

Let $(b+a) = n^{r-1}$ and $(b-a) = n$. Then:

$$b = \frac{n}{2}(n^{r-2}+1) \quad \text{and} \quad a = \frac{n}{2}(n^{r-2}-1) .$$

The bth odd number will then be $n^{r-1}+n-1$ and the $(a+1)$ st odd number will be $n^{r-1}-n+1$. Therefore n^r can be represented by the sum of all the

odd numbers included within this range.

We thus obtain:

$$\text{(i)} \qquad n^r = \sum_{t=1}^{n} \{2t + (n^{r-1} - n - 1)\} .$$

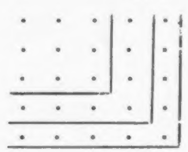
The validity of (i) is easily checked, since it equals

$$2\left(\frac{n^2}{2} + \frac{n}{2}\right) + (n^{r-1} - n - 1)n = n^2 + n + n^r - n^2 - n = n^r .$$

Since the summation runs from 1 to n there will of course be precisely n odd numbers in the set. It can be seen further that the expression after the summation sign must represent an odd number, since it is of the form $2t + m$, m odd, since $n^{r-1} - n - 1$ is odd for n odd or even.

It can now also be seen that the expression of 2^4 as $7+9$, which is rejected on page 134 of Sugai's article, is legitimate, and does not in fact violate the nature of gnomons as suggested. It can be shown thus:

7 + 9



More fully, n^4 can be shown as follows:

$$1^4 = 1 .$$

$$2^4 = 7 + 9 .$$

$$3^4 = 25 + 27 + 29 .$$

$$4^4 = 61 + 63 + 65 + 67 .$$

. . . . .

. . . . .

. . . . .

In all cases, diagonal and vertical elements are polynomials.

But the situation as shown above may be generalized. n^r may be represented not only as n consecutive odd numbers, but as $n^2, n^3, \cdots, n^p$ consecutive odd numbers, where $p \leq r/2$. Using the same technique as previously, we may put:

$$(b+a)(b-a) = n^r = n^p \cdot n^{r-p} .$$

From which we obtain:

$$b = \frac{n^p}{2}(n^{r-2p} + 1) \quad \text{and} \quad a = \frac{n^p}{2}(n^{r-2p} - 1) .$$

So that the bth odd number will be $n^{r-p}+n^p-1$ and the $(a+1)$ st odd number will be $n^{r-p}-n^p+1$. And, again, n^r can be represented by the sum of all the n^p consecutive odd numbers included within this range.

We thus have:

$$\text{(ii)} \qquad n^r = \sum_{t=1}^{n^p} \{2t+(n^{r-p}-n^p-1)\}\ .,$$

which may be checked as previously.

n^2 can only be represented by an "overlapping" series of odd numbers, in each case commencing with 1. If r is even, we may put $n^r = n^{2s}$. All cases of this nature can be expressed as $(b+a)(b-a) = n^s \cdot n^s$. From which $b = n^s$, $a = 0$. So that the series of n^s consecutive odd numbers starts with 1 (the $(a+1)$ st) for each n. This is, of course, in addition to representation by sets of n, n^2, $\cdots$, n^{s-1} consecutive odd numbers, none of which "overlaps", where $s > 1$ (i. e. r an even number greater than 2).

Example. $r = 2s = 6$.

n	n^6	n gnomons	n^2 gnomons	n^3 gnomons
1	1	1	1	1
2	64	$31+33$	$13+15+17+19$	$1+3+5+7+9+11+13+15$
3	729	$241+243+245$	$73+75+\cdots+89$	$1+3+5+\cdots+53$
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Pine Tree Cottage
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BOOK REVIEWS

Similarity and Dimensional Methods in Mechanics. By L. I. Sedov. Translation edited by M. Holt. Translation by M. Friedmann. Academic Press, Inc., New York, 1960, 364 pages. $14.00.

This is a translation of a Russian book which has gone through four editions since its original publication in 1943. There is an initial chapter on general dimensional theory, a second chapter which surveys a dozen or so of its applications, and a third chapter with a very full treatment of viscous fluid theory and statistical theories of turbulence. Then there are two chapters on unsteady gas motions and astrophysical problems (explosions, flame propagation, stellar pulsations, and stellar flare ups). These last chapters constitute the major part of the book and its most unique feature, since they represent a unified treatment of original research which is not otherwise available.

The treatment emphasizes the use of dimensional analysis in the solution of the differential equations rather than as a technique for correlating experimental data. Dimensional and similarity arguments are vigorously and systematically employed to reduce the number of variables, to reduce the order of the differential equations, and to establish the existence and geometric properties of self-similar solutions.

The book may not appeal to mathematicians whose interests lie in a more rigorous analysis of the fundamentals of dimensional analysis and its interpretation by transformation theory or group-theoretic methods. Actually the presentation of the basic ideas of dimensional analysis seems rather cursory and disorganized almost as though it was added as an afterthought. Perhaps some of the trouble lies in the translation which appears too mechanical and lacks fluency. However the book can certainly be recommended to physicists who are interested in the particular fields of fluid mechanics or in methods of solving differential equations.

George Chertock

A Brief Course in Analytics. By M. A. Hill, Jr., and J. B. Linker. Holt, Rinehart and Winston, Inc., New York, third edition, 1960, VIII + 232. $3.90.

This book is a second revision of a standard treatment of analytic geometry. The writing is clear and the problems are plentiful and well selected. The organization of material, diagrams, and typography are good.

One is inclined to ask, "What is the place of analytic geometry in the mathematics curriculum of today?" One can give an elegant treatment of the subject which is excellent for students majoring in mathematics. Such a treatment is probably too extended for the crowded curriculum of the engineering student. This book comes close to finding a happy medium which

is suitable for both groups.

This reviewer believes that there is merit in using a standard notation and terminology for mathematics. In discussing the idea of necessary and sufficient conditions, most writers use the if and only if type of sentence structure. In their treatment of the locus concept these authors talked about a curve as the locus of all points, and no others, whose coordinates satisfy a given equation. The all, and no others sentence does not seem to be superior to the if and only if sentence, and it is not the standard way in which the student will find the necessary and sufficient condition expressed in more advanced books.

The treatment of curves in polar coordinates seems rather brief. From the various ways to handle the algebraic sign of the radical in the formula for the distance from a point to a line, these authors use the one which this reviewer feels is the most satisfactory. A final chapter covers an introduction to the analytic geometry of three dimensions.

This book would be quite satisfactory for a three hour, one semester course.

R. E. Horton

Differential Equations. By Ralph Palmer Agnew. Second Edition. New York, McGraw-Hill, 1960, ix + 485 pp. $7.50.

This is an augmented and enlarged version of the author's text (and reference book) by the same title, which appeared in a first edition in 1942.

The principal difference between the first and the second editions is the inclusion of additional problems even unto the third and fourth decimal places and beyond. These problems present a challenge to the student as well as embodying new programs for obtaining solutions of differential equations. The text has proceeded in both editions, a long way from the old fashioned differential equation text which presented tricks and cook book recipes (including keys) for solving equations without a glimmer of the student knowing anything about what he was doing and without finding out what constitutes a solution. There is much "small talk" in the text (not intended to be taken too seriously) such as comments which an interesting teacher would use to spice his daily class meetings.

In this day of cutthroat competition in science with other nations such as Russia and China, this text will be very helpful to stimulate the average and the superior student to equip himself to do mathematics and to make use of it in the service of man. Furthermore, the pace suggested by the author in the Assignment Chart for a 40 lesson semester will keep busy the most pernickitty talented student who has the yen to leap ahead.

In the new chapter on Laplace Transforms, the author puts in a justifiable plug for the Heaviside Operators, but does nothing much further

about it. He leans quite "heavily" instead on the classic D-operator which could be readily extended to the Heaviside operator at its best.

Chapter 13 is in the nature of an appendix on basic elementary calculus. Another arrangement could have been obtained by renumbering and calling this chapter 1. The best reason of many which can be given for doing this, is that the concepts, limits, derivatives, integrals, introduced afford the best kind of preliminary introduction to the methods, uses and meanings of differential equations.

The last chapters 14, 15, 16 are excellent "lures" to entice the student of the sciences (of which mathematics is one) to follow the siren of deeper and broader study of calculus and equations of the partial differential, integral and integro-differential type and systems, of such, etc., etc., and so on.

H. J. Ettlinger

Complex Variables and Applications. By Ruel V. Churchill. Second Edition, New York, McGraw-Hill, 1960, ix + 297 pp. $6.75.

The first edition* was published in 1948. This second edition is an extension and a revision of the first "with greater attention to sound logical procedures and clarity." The foregoing quote is from the author's preface to the revised edition. Again this text is designed for a one semester course, but it is easily perceived that there is sufficient material with excellent problems for a year's course and beyond.

This book like all of Churchill's texts is valuable, nay indispensable, for working courses in analysis. The direction of development is for the introduction of operators with ordered number pairs, (complex numbers) at an earlier level in secondary teaching of mathematics. The further development of ideas involved in basic concepts (theory, so-called) is to be commended for a thoroughgoing understanding of the "use" of complex numbers in the broadest sense.

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*The first edition carried the title, *Introduction to Complex Variables and Applications*, and was reviewed in this Mathematics Magazine, Vol. XXIII, No. 3, 1950, pp. 153-4, by Professor Robert E. Greenwood.

H. J. Ettlinger

BOOKS RECEIVED FOR REVIEW

Elementary Concepts of Sets. By Edith J. Woodward and Roderick C. McLennan. Holt, Rinehart and Winston Inc., New York, 1960, v + 50 pp. $.60.

Applied Boolean Algebra an Elementary Introduction. By Franz E. Hohn. The MacMillan Company, New York, 1960, xx + 139 pp. $2.50.

Introduction to Linear Programming. By Walter W. Garvin. McGraw-Hill Book Company, New York, 1960, xiv + 281 pp. $8.75.

Fundamentals of Mathematics. By E. P. Vance. Addison-Wesley Publishing Company, Reading, Massachusetts, 1960, x + 469 pp. $7.50.

BISECTION OF YIN AND OF YANG

C. W. Trigg

The monad, a configuration in which a circle is bisected by two equal semicircles, recurrently becomes the subject of discussion. One problem requires that each of the congruent parts, sometimes called *Yin* and *Yang*, of the circle be bisected by a single line. Five methods to accomplish this are offered here, with the restriction that the bisecting line can be constructed by Euclidean means. It is assumed that the diameter on which the semicircles are drawn is determinable.

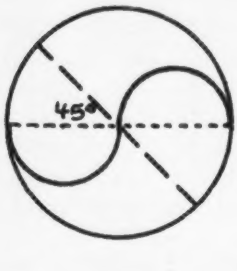


Figure 1

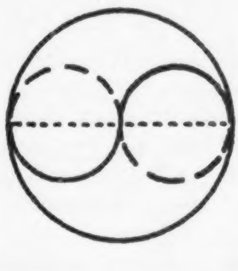


Figure 2

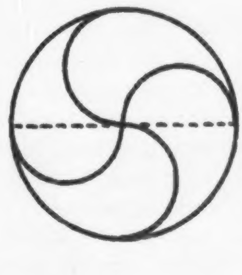


Figure 3

Each semicircle with its diameter contains one-eighth of the area of the large circle ($\pi R^2/8$). Also, the area of a 45°-sector of the large circle is $\pi R^2/8$. Hence, a *straight* line through the center of the circle making a 45°-angle with the diameters of the semicircles accomplishes the bisection (Figure 1).

If the monad be reflected onto itself about the diameter of the semicircles, two circles with radii $R/2$ are formed (Figure 2) and the large circle is divided into four equal areas. The division is accomplished by a *continuous* curved line composed of two semicircles.

If the monad be rotated 90° onto itself about its center, again the circle is quadrisected by a curved line composed of two semicircles (Figure 3). In this case, the four parts are congruent. In fact, $k-1$ successive rotations through angles of $180°/k$ will divide the circle into $2k$ congruent

parts.

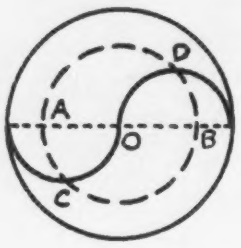


Figure 4

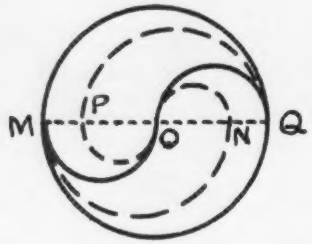


Figure 5

A circle concentric to the large circle and with radius $R/\sqrt{2}$ bisects *Yin* and *Yang*. This is evident since the areas AOC and DOB in Figure 4 are congruent, and the area of half the smaller circle is $\pi R^2/4$.

The continuous curved line $MNOPQ$ in Figure 5, composed of semicircles with radii $(\sqrt{5}+1)R/4$ and $(\sqrt{5}-1)R/4$, also bisects *Yin* and *Yang*. This follows since

$$\tfrac{1}{2}\pi[R(\sqrt{5}+1)/4]^2 + \tfrac{1}{2}\pi[R(\sqrt{5}-1)/4]^2 - \tfrac{1}{2}\pi[R/2]^2 = \pi R^2/4 .$$

Los Angeles City College

ENVELOPES AND NODES

If the finding is essential
Of a singular solution
For equations differential,
Let me sketch its execution.

First obtain some ordinary
Members of the family
Of solutions – oh, not very
Many, maybe twenty three.

Find their curves by careful plotting.
Ink them quickly and you ought,
From the points requiring blotting,
To obtain the locus sought.

Marlow Sholander

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbook should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

425. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

If $n-1$ and $n+1$ are twin prime numbers, prove that $3\phi(n) \leqq n$ where ϕ denotes Euler's ϕ-function.

426. *Proposed by Dmitri Thoro, San Jose State College, California.*

Find the number N_p of non-singular matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose elements belong to the residue class modulo p, where p is a prime.

427. *Proposed by D. Moody Bailey, Princeton, West Virginia.*

P is any point in the plane of a triangle ABC through which cevians from B and C are drawn meeting sides CA and AB at points E and F respectively. M is the midpoint of BC and line MP meets CA at N and AB at O. EF extended meets BC at G and a line through B parallel to AG meets CF at H. Show that HO is parallel to CA.

428. *Proposed by Murray S. Klamkin, AVCO, Wilmington, Massachusetts.*

The number $N = 142{,}857$ has the property that $2N$, $3N$, $4N$, $5N$, and $6N$ are all permutations of N. Does there exist a number M such that $2M$, $3M$, $4M$, $5M$, $6M$, and $7M$ are all permutations of M?

429. *Proposed by M. S. Krick, Albright College, Pennsylvania*

Verify that

$$\sum_{k=1}^{n} 1/k\ [1+(-1)^k\tbinom{n}{k}] = 0\ .$$

430. *Proposed by Leon Bankoff, Los Angeles, California.*

At a point P on the latus rectum of a parabola, a perpendicular to the latus rectum is erected, cutting the curve at Q. Show that PQ is half the harmonic mean of AP and PB.

431. *Proposed by William Squire, Southwest Research Institute, San Antonio, Texas.*

Given a rectangular array of numbers

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & N \\ 2 & 3 & 4 & 5 & \cdots & N+1 \\ 3 & 4 & 5 & 6 & \cdots & N+2 \\ \vdots & & & & & \vdots \\ M & & & & & M+N-1 \end{array}$$

How many paths are there going in correct numerical order from 1 to $M+N-1$?

SOLUTIONS

Late Solutions

401. *William Squire, Southwest Research Institute, San Antonio, Texas.*
382, 397, 399, 400, 401, 402. *Josef Andersson, Vaxholm, Sweden.*

Hobson's Choice

404. [March 1960] *Proposed by Barney Bissinger, Lebanon Valley College, Pennsylvania.*

On page 89 in Hobson's "Treatise on Plane Trigonometry", 2nd edition, 1897, exercise 1 asks in part to prove that

$$\frac{\sin n\alpha}{\sin\alpha} = 2[\cos(n-1)\alpha + \cos(n-3)\alpha + \cos(n-5)\alpha + \cdots].$$

Show that this formula holds only for even n and find the correct formula for odd n.

I. *Solution by M. Morduchow, Polytechnic Institute of Brooklyn.*

For even n, note that $F \equiv [\cos\alpha + \cos 3\alpha + \cdots + \cos(n-1)\alpha]$ is the real part of $s \equiv [e^{i\alpha} + e^{3i\alpha} + \cdots + e^{(n-1)i\alpha}]$. But s is a geometric series with common ratio $e^{2i\alpha}$. Hence,

$$s = \frac{e^{i\alpha}(e^{2i\alpha})^{n/2} - e^{i\alpha}}{e^{2i\alpha}-1} = \frac{e^{i\alpha}(e^{in\alpha}-1)}{e^{2i\alpha}-1} = \frac{e^{in\alpha}-1}{e^{i\alpha}-e^{-i\alpha}}$$

$$= \frac{\cos n\alpha + i\sin n\alpha - 1}{2i\sin\alpha}.$$

$$\therefore \quad F = \frac{\sin n\alpha}{2 \sin \alpha},$$

and the formula to be proven follows.

For odd n, we see immediately that the formula cannot hold since it is invalid for $n = 1$. For odd n, we now note that

$$G \equiv [1 + \cos 2\alpha + \cos 4\alpha + \cdots + \cos (n-1)\alpha]$$

is the real part of

$$T \equiv [1 + e^{2i\alpha} + e^{4i\alpha} + \cdots + e^{(n-1)i\alpha}] .$$

But

$$T = \frac{(e^{2i\alpha})^{((n+1)/2)} - 1}{e^{2i\alpha} - 1} = \frac{e^{(n+1)i\alpha} - 1}{e^{2i\alpha} - 1} = \frac{e^{in\alpha} - e^{-i\alpha}}{e^{i\alpha} - e^{-i\alpha}}$$

$$= \frac{(\cos n\alpha - \cos \alpha) + i(\sin n\alpha + \sin \alpha)}{2i \sin \alpha}.$$

Hence,

$$\frac{\sin n\alpha + \sin \alpha}{2 \sin \alpha} = G ,$$

and the correct formula is

$$\frac{\sin n\alpha}{\sin \alpha} = 2[\cos (n-1)\alpha + \cos (n-3)\alpha + \cdots + 1] - 1 .$$

(Note: From the above analysis, by taking imaginary parts instead of real parts, it also follows that for even n ($\geqq 2$):

$$\frac{1 - \cos n\alpha}{\sin \alpha} = 2[\sin (n-1)\alpha + \sin (n-3)\alpha + \cdots + \sin \alpha] .$$

for odd n ($\geqq 3$):

$$\frac{\cos \alpha - \cos n\alpha}{\sin \alpha} = 2[\sin (n-1)\alpha + \sin (n-3)\alpha + \cdots + \sin 2\alpha]\Big) .$$

II. *Solution by J. L. Brown, Jr., Ordinance Research Laboratory, University Park, Pennsylvania.*

Multiply

$$\cos \alpha + \cos 3\alpha + \cdots + \cos (2n-1)\alpha$$

by $2 \sin \alpha$ to obtain the product

$$\sin 2\alpha + [\sin 4\alpha - \sin 2\alpha] + \cdots + [\sin 2n\alpha - \sin (2n-2)\alpha]$$

which telescopes to $\sin 2n\alpha$. Thus

$$\frac{\sin 2n}{\sin \alpha} = 2 \sum_{k=0}^{n-1} \cos (2k+1)\alpha$$

or, equivalently

$$\frac{\sin 2n\alpha}{\sin\alpha} = 2\sum_{K=1,3,5,\cdots}^{2n-1} \cos(2n-K)\alpha ,$$

as required. Similarly, multiplying

$$\tfrac{1}{2} + \cos 2\alpha + \cos 4\alpha + \cdots + \cos 2n\alpha$$

by $2\sin\alpha$, one obtains

$$\frac{\sin(2n+1)\alpha}{\sin\alpha} = 1 + 2\sum_{K=0}^{n} \cos 2K\alpha ,$$

or, alternatively

$$\frac{\sin(2n+1)\alpha}{\sin\alpha} = 1 + 2\sum_{K=0,2,4,\cdots}^{2n} \cos(2n-K)\alpha .$$

NOTE : Since the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ are defined by

$$T_n(\cos\theta) = \cos n\theta \quad \text{and} \quad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} ,$$

these trigonometric formulas are equivalent to the identities

$$U_{2n-1}(x) = 2\sum_{K=0}^{n-1} T_{2k+1}(x)$$

and

$$U_{2n}(x) = 1 + 2\sum_{K=0}^{n} T_{2k}(x) ,$$

which are well-known, e.g. "*Higher Transcendental Functions*," Vol. 2, Ed. by A. Erdelyi, McGraw-Hill Book Co., Inc., 1953, p. 187, equation (40). [In equation (40) of this reference, the right hand side of the first equality whould read "$-\tfrac{1}{2} + \tfrac{1}{2}U_{2n}(x)$."]

III. *Solution by Lawrence A. Ringenberg, Eastern Illinois University.* From the identity

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

we get

$$\sin(n+2)\alpha - \sin n\alpha = 2\cos(n+1)\alpha\sin\alpha ,$$

$$\frac{\sin (n+2)\alpha}{\sin \alpha} = 2\cos (n+1)\alpha + \frac{\sin n\alpha}{\sin \alpha} .$$

Since

$$\frac{\sin \alpha}{\sin \alpha} = 1 \quad \text{and} \quad \frac{\sin 2\alpha}{\sin \alpha} = 2\cos \alpha ,$$

it follows by induction that the formula stated in the problem holds for n even and that the correct formula for odd n is

$$\frac{\sin n\alpha}{\sin \alpha} = 2[\cos (n-1)\alpha + \cos (n-3)\alpha + \cdots + \cos 2\alpha] + 1 .$$

IV. *Solution by Dmitri Thoro, San Jose State College, California.* Let

$$u(x) = \cos (n-2x-1)\alpha .$$

From the Calculus of Finite Differences,

$$\Delta^{-1}u(x) = \frac{\sin (2x-n)\alpha}{2\sin \alpha} .$$

Thus

$$\sum_{x=0}^{n-1} u(x) = \frac{\sin n\alpha}{\sin \alpha} .$$

But

$$\sum_{x=0}^{n-1} u(x) = \begin{cases} 2\sum_{x=0}^{k-1} u(x) & \text{when } n = 2k . \\ 1 + \sum_{x=0}^{k-1} u(x) & \text{when } n = 2k+1 . \end{cases}$$

Hence

$$\frac{\sin n\alpha}{\sin \alpha} = \begin{cases} 2[\cos (n-1)\alpha + \cos (n-3)\alpha + \cdots + \cos \alpha] , & n \text{ even.} \\ 1 + 2[\cos (n-1)\alpha + \cos (n-3)\alpha + \cdots + \cos 2\alpha] , & n \text{ odd.} \end{cases}$$

Also solved by J. W. Clawson, Collegeville, Pennsylvania; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; Charles F. Pinzka, University of Cincinnati; William Squire, Southwest Research Institute, San Antonio, Texas; P. Vasic, University of Belgrade, Yugoslavia; and the proposer.

A Pedal Triangle

406. [March 1960] *Proposed by M. N. Gopalan, Mysore City, India.*

If α, β, and γ are the sides of the pedal triangle of a triangle ABC, prove that:

$$\frac{\alpha+\beta+\gamma}{a+b+c} = [\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}-1] .$$

Solution by J. W. Clawson, Collegeville, Pennsylvania.

Let the feet of the altitudes from A, B, C to the opposite sides be D, E, F. Then, applying the law of sines to triangle AEF,

$$\frac{\alpha}{\sin A} = \frac{b\cos A}{\sin B},$$

since $\measuredangle AEF = \measuredangle AHF = \Pi/2 - \measuredangle BAD = \measuredangle B$. But $b = 2R\cdot\sin B$, where R is the radius of the circumcircle. Hence $\alpha = R\cdot\sin 2A$.

$$\begin{aligned}\frac{\alpha+\beta+\gamma}{a+b+c} &= \frac{\sin 2A + \sin 2B + \sin(2\Pi - 2A - 2B)}{2[\sin A + \sin B + \sin(\Pi - A - B)]}\\ &= \frac{2\cdot\sin(A+B)[\cos(A-B) - \cos(A+B)]}{4\cdot\sin\dfrac{A+B}{2}[\cos\dfrac{A-B}{2} + \cos\dfrac{A+B}{2}]}\\ &= \frac{\cos\dfrac{A+B}{2}[2\cdot\sin A\cdot\sin B]}{2\cos\dfrac{A}{2}\cdot\cos\dfrac{B}{2}}\\ &= 4\cdot\cos\frac{A+B}{2}\cdot\sin\frac{A}{2}\cdot\sin\frac{B}{2}\end{aligned}$$

$$\begin{aligned}\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}-1 &= \cos A + \cos B + \cos C - 1\\ &= \cos A + \cos B - [1+\cos(A+B)]\\ &= 2\cdot\cos\frac{A+B}{2}\cdot[\cos\frac{A-B}{2} - \cos\frac{A+B}{2}]\\ &= 4\cdot\cos\frac{A+B}{2}\cdot\sin\frac{A}{2}\cdot\sin\frac{B}{2} .\end{aligned}$$

Comments by C. W. Trigg, Los Angeles City College.

The pedal triangle referred to is the pedal triangle of the orthocenter, or the orthic triangle. The following theorems are quoted from N. A. Court, *College Geometry*, Johnson Publishing Co. (1925),

Page 89. "The sum of the ratios of the sides of the orthic triangle to the corresponding sides of the given triangle is equal to the ratio of the sum of the circumradius and the inradius of the given triangle

to the circumradius of the same triangle."

Thus,

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=\frac{R+r}{R}=1+\frac{r}{R}\,.$$

Page 89. "The perimeter of the orthic triangle is equal to the double area of the given triangle divided by the circumradius of the given triangle."

Page 71. "The inradius of a triangle is equal to the area of the triangle divided by half the perimeter."

Thus,

$$\alpha+\beta+\gamma=\frac{2\Delta}{R}=\frac{r(a+b+c)}{R}\,.$$

The given relationship follows immediately.

Also solved by D. Moody Bailey, Princeton, West Virginia; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; W. Moser, University of Manitoba; Charles F. Pinzka, University of Cincinnati; P. Vasic, University of Yugoslavia; and the proposer.

Resistance In A Cube

407 [March 1960] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

The twelve edges of a cube are made of wires of one ohm resistance each. The cube is inserted into an electrical circuit by:

a) two adjacent vertices,

b) two opposite vertices of a face,

c) two opposite vertices of the cube.

Which offers the least resistance?

Solution by C. W. Trigg, Los Angeles City College.

It may be inferred that the least resistance occurs in (a) since there is a single-edge connector between the terminals. For confirmation:

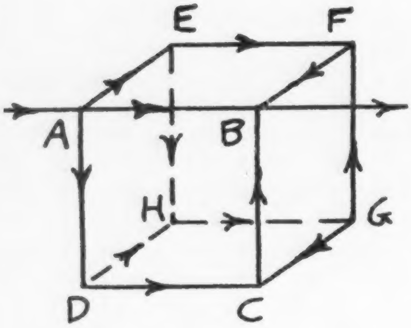
In the figures, the direction of current flow is shown in each case. Below each cube a schematic diagram is shown wherein corners at the same potential, as determined by symmetry, are represented by the same point. Each situation is thus reduced to the simple case of repeated application of the laws of parallel circuits. So:

A) $$1/R = 1/r + 1/\{r/2 + 1/[2/r + 1/(r/2 + r + r/2)] + r/2\}\,,$$

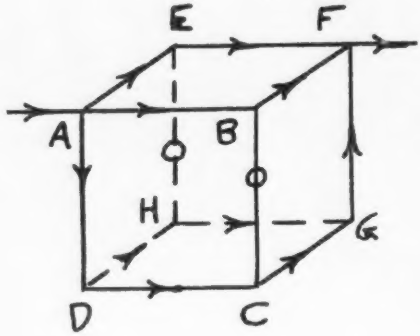
whence $R = 7/12\ r$, where R is the resistance of the cube and r is 1 ohm.

B) E, B, H and C are at the same potential, so

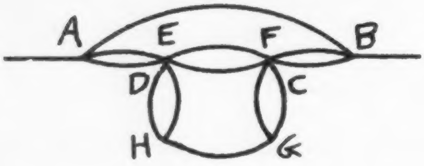
$$R = 2/\{1/r + 1/r + 1/[r + r/2]\}\quad\text{or}\quad 3r/4\,.$$

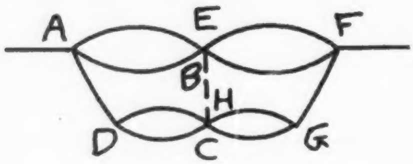


(a)



(b)





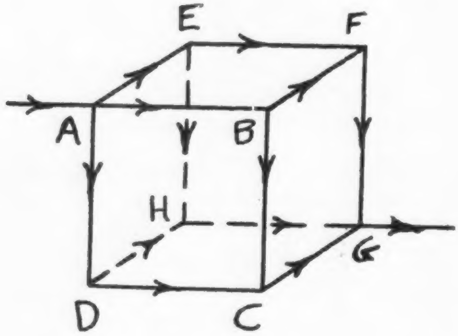
C) $R = r/3 + r/6 + r/3$

or $5r/6$.

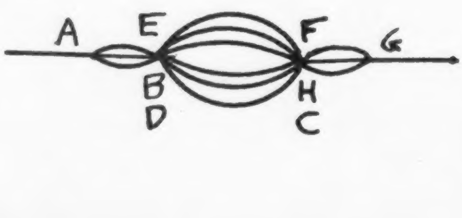
Cases (a) and (c) are solved on pages 277-279 of *Magnetism and Electricity* by E. E. Brooks and A. W. Poyser, Longmans, Green and Co. (1920).

Case (c) is Quickie 32, MATHEMATICS MAGAZINE, March 1951, November 1959.

Also solved by Charles F. Pinzka, University of Cincinnati; and the proposer (partially). One incorrect solution was received.



(c)



Circumscribed Ellipses

408 [March 1960] *Proposed by M. S. Klamkin, AVCO, Lawrence, Massachusetts.*

Three congruent ellipses are mutually tangent symmetrically. Determine the radius of the circumcircle.

Solution by Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania.

Let C denote the intersection of the common tangents to the three ellipses. Extend the major axes of two of the ellipses and denote the point of intersection by P. Let T denote the point of tangency of these two ellipses. Denote the center of one of these ellipses by H, then angle $HPC = 30°$. Let F and G be the foci of the ellipse with center H with points in the order F, H, G, P. Let I, J be the foci of the other ellipse with J between I and P.

If a, b, c have their usual meaning, then, by the law of cosines

$$(2a)^2 = \overline{GP}^2 + \overline{IP}^2 - \overline{GP} \cdot \overline{IP}\,.$$

Using $\overline{IP} = \overline{GP} + 2c$, and solving for $\overline{GP} + c$, we obtain

$$\overline{GP} + c = \sqrt{4b^2 + c^2} = \sqrt{a^2 + 3b^2}\,.$$

From triangle CHP,

$$\overline{HC} = \frac{c + \overline{GP}}{\sqrt{3}} = \sqrt{\frac{a^2 + 3b^2}{3}}\,.$$

Selecting coordinate axes with origin at H, with positive x-axis in direction HP and positive y-axis in direction HC, then the ellipse with center H has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and point C has the coordinates

$$\left(0,\ \sqrt{\frac{a^2 + 3b^2}{3}}\right).$$

By means of the calculus, it can be shown that the poir[illegible] ellipse at maximum distance from C have coordinates

$$\left(\pm a\{1 - [b^2/c^4]\frac{(a^2 + 3b^2)}{3}\},\ -\frac{b^2}{c^2}\sqrt{\frac{a^2 + 3b^2}{3}}\right).$$

The desired radius is then computed by the distance formula, and is

$$\frac{2a^2}{\sqrt{3}\,c}\,.$$

Also solved by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; Walter B. Carver, Ithaca, New York; J. W. Clawson, Collegeville, Pennsylvania; and Charles F. Pinzka, University of Cincinnati.

A Factorial Congruence

409 [March 1960] *Proposed by D. S. Mitrinovich, University of Belgrade, Yugoslavia.*

Prove

$$(kn)! \equiv 0 \left[\bmod \prod_{r=0}^{n-1} (n+r)\right] \quad \text{where} \quad n \geqq k .$$

Solution by Dale Woods, State Teachers College, Kirksville, Missouri.

If $k = 1$ we must have $n! = M(n)(n+1)(n+2)(n+3)\cdots(2n-1)$ for $M \geqq 1$, which is impossible.

If $k \geqq 2$ then $(kn)! = M(n-1)!\,(n)(n+1)(n+2)(n+3)\cdots(2n-1)$ for $M \geqq 1$ therefore

$$(kn)! \equiv 0 \left[\bmod \prod_{r=0}^{n-1} (n+r)\right] \quad \text{where } n \geqq k \geqq 2 .$$

Also solved by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey; Charles F. Pinzka, University of Cincinnati; William Squire, Southwest Research Institute, San Antonio, Texas; C. W. Trigg, Los Angeles City College; and P. Vasic, University of Belgrade, Yugoslavia.

A Functional Relation

410 [March 1960] *Proposed by Robert W. Kilmoyer, Jr., Lebanon Valley College, Pennsylvania.*

The functions $x(t)$ and $y(t)$ exist along with their derivatives $x'(t)$ and $y'(t)$ respectively. If $x(r^2+s^2) = y(r)y(s)$ where r and s are any independent variables, find $x(t)$ and $y(t)$.

Solution by Charles F. Pinzka, University of Cincinnati.

Rejecting the trivial solution $x(t) = y(t) = 0$, we have, differentiating successively with respect to r and s,

$$2r\,x'(r^2+s^2) = y'(r)y(s) \quad \text{and} \quad 2s\,x'(r^2+s^2) = y(r)y'(s) .$$

This gives

$$\frac{r}{s} = \frac{y'(r)y(s)}{y(r)y'(s)}, \quad \text{or} \quad \frac{y'(r)}{ry(r)} = \frac{y'(s)}{sy(s)} = c \text{ , a constant.}$$

Thus $y(t) = e^{c_1 t^2}$, from which follows $x(t) = e^{c_1 t}$, where $t \geqq 0'$

Also solved by Dermott A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; Huseyin Demir, Kandilli, Eregli, Kdz, Turkey;

Gaines B. Lang, University of Florida; M. Morduchow, Polytechnic Institute of Brooklyn; Lawrence A. Ringenberg, Eastern Illinois University; William Squire, Southwest Research Institute, San Antonio, Texas; and the proposer.

INTEGER SOLUTIONS OF $AB/(A+B)$

C. W. Trigg

There are twenty-three two-digit integers which are exactly divisible by the sums of their digits. So, every integer from 2 to 10 inclusive can be represented by $AB/(A+B)$, some in more than one way, and 10 can be represented in nine ways. Thus

$AB/(A+B)$	AB								
2	18								
3	27								
4	12	24	36	48					
5	45								
6	54								
7	21	42	63	84					
8	72								
9	81								
10	10	20	30	40	50	60	70	80	90

Los Angeles City College

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 270. The triangle with sides a, b, and c is equilateral if

$$a^2+b^2+c^2=ab+bc+ca\ .$$

[*Submitted by C. W. Trigg*]

Q 271. If two perpendicular lines are drawn in the plane of a square, the segment intercepted by a pair of opposite sides of the square on one of the two lines is equal to the segment which the other pair of opposite sides intercepts on the other given line. [*Submitted by C. W. Trigg*]

TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase or idea rather than upon a mathematical routine. Send us your favorite trickies.

T 39. Find the sum of

$$S=\sum_{n=1}^{\infty} 1/p_n$$

where p_n is the nth prime in the sequence n^5+n+1 . [*Submitted by M. S. Klamkin*]

T 40. Write the next three digits in the sequence 6, 8, 5, 8, 4, 0, 7, 3, 4, 6. 4, ... [*Submitted by C. W. Trigg*]

FALSIES

A falsie is a problem for which a correct solution is obtained by illegal operations, or an incorrect result is securred by apparently legal processes. For each of the following falsies can you offer an explanation? Send us your favorite falsies for publication.

F 18. We have

$$e^{ix}=\cos x+i\sin x\ .$$

Thus

$$e^{\pi i}=-1 \quad\text{and}\quad e^{3\pi i}=-1$$

so that

$$e^{\pi i}=e^{3\pi i}\ .$$

Taking the natural logarithm of both sides we obtain

$$\pi i = 3\pi i \quad \text{or} \quad 3 = 1 \,.$$

[*Submitted by Paul H. Yearout*]

F 19. Given

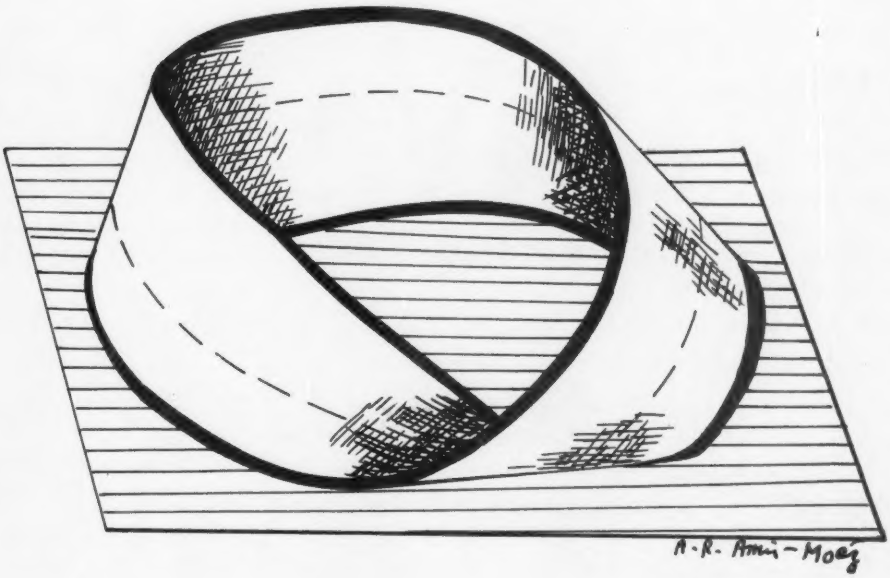
$$F(x) = \sin^2 x \,.$$

Then

$$F(\pi/2 - x) = \sin^2(\pi/2 - x) = \sin^2 \pi/2 - \sin^2 x = 1 - \sin^2 x = 1 - F(x) \,.$$

[*Submitted by Roger Osborn*]

(Answers to Quickies, Solutions to Trickies, and Explanations of Falsies are on page 122.)



Möbius strip

(Answers to Quickies, Solutions to Trickies, and Explanations of Falsies which appear on pages 120-121)

ANSWERS

A 270. The expression may be written as

$$-(a-b)^2 = (c-a)(c-b) .$$

This is impossible if $c > b > a$. Otherwise the expression may be written as

$$\begin{vmatrix} a & b & 1 \\ b & c & 1 \\ c & a & 1 \end{vmatrix} = 0 .$$

Then either

$$a = b = c = 1 , \quad \text{or} \quad \frac{a}{b} = \frac{b}{c} = \frac{c}{a} ,$$

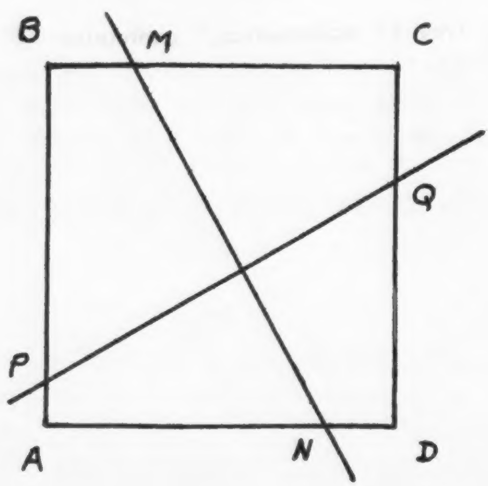
so

$$b^2 = ac \quad \text{and} \quad a^2 = bc ,$$

whence

$$b^3 = c^3 \quad \text{so} \quad a = b = c .$$

A 271. Leave PQ in position in the plane and rotate the remainder of the figure through 90° about the center of the square. The square will then coincide with its original position and MN (formerly perpendicular to PQ) will now be parallel to PQ and hence equal to PQ.



SOLUTIONS

S 39. Since

$$(n^5+n+1) = (n^2+n+1)(n^3-n^2+1)$$

there is only one prime $p_1 = 3$. Whence $S = 1/3$.

S 40. The sequence was obtained by subtracting $\pi = 3.14159265358979\cdots$ from $9.999\cdots$, so the next three terms are 1, 0, 2.

EXPLANATIONS

E 18. The ln(−1) and the ln(−1) may lie on two different Reimann surfaces.

E 19. The correct solution is given by

$$\begin{aligned}\sin^2(\pi/2 - x) &= (\sin\pi/2\cos x - \cos\pi/2\sin x)^2 .\\ &= \cos^2 x\\ &= 1-\sin^2 x\end{aligned}$$

so

$$F(\pi/2-x) = 1-F(x) .$$

This falsie is akin to the proof that $\frac{\sin x}{n} = 6$ is an identity by cancelling the n in the numerator with the one in the denominator leaving $six = 6$, which surely is identically true.

(*On the Digital Roots of Perfect Numbers,* continued from page 100)

Tabulate:

Digital Root of:

n	2^n	2^n-1	2^{n-1}	$2^{n-1}(2^n-1)$
1	2	1	1	1
2	4	3	2	6
3	8	7	4	1
4	7	6	8	3
5	5	4	7	1
6	1	9	5	9
7	2	1	1	1
8	4	3	2	6
9	8	7	4	1
10	7	6	8	3
11	5	4	7	1
12	1	9	5	9

The digital roots of the formula

$$2^{n-1}(2^n-1)$$

form the sequence

$$1,\ 6,\ 1,\ 3,\ 1,\ 9.$$

When n is odd, the digital root is 1.

But when n is even, the digital roots of 2^n-1 form the sequence

$$3,\ 6,\ 9,\ 3,\ 6,\ 9.$$

This indicates that when n is even, 2^n-1 is divisible by 3. Hence n must be odd before

$$2^{n-1}(2^n-1)$$

can form a perfect number. And its digital root must be 1.

Sweeny
Texas

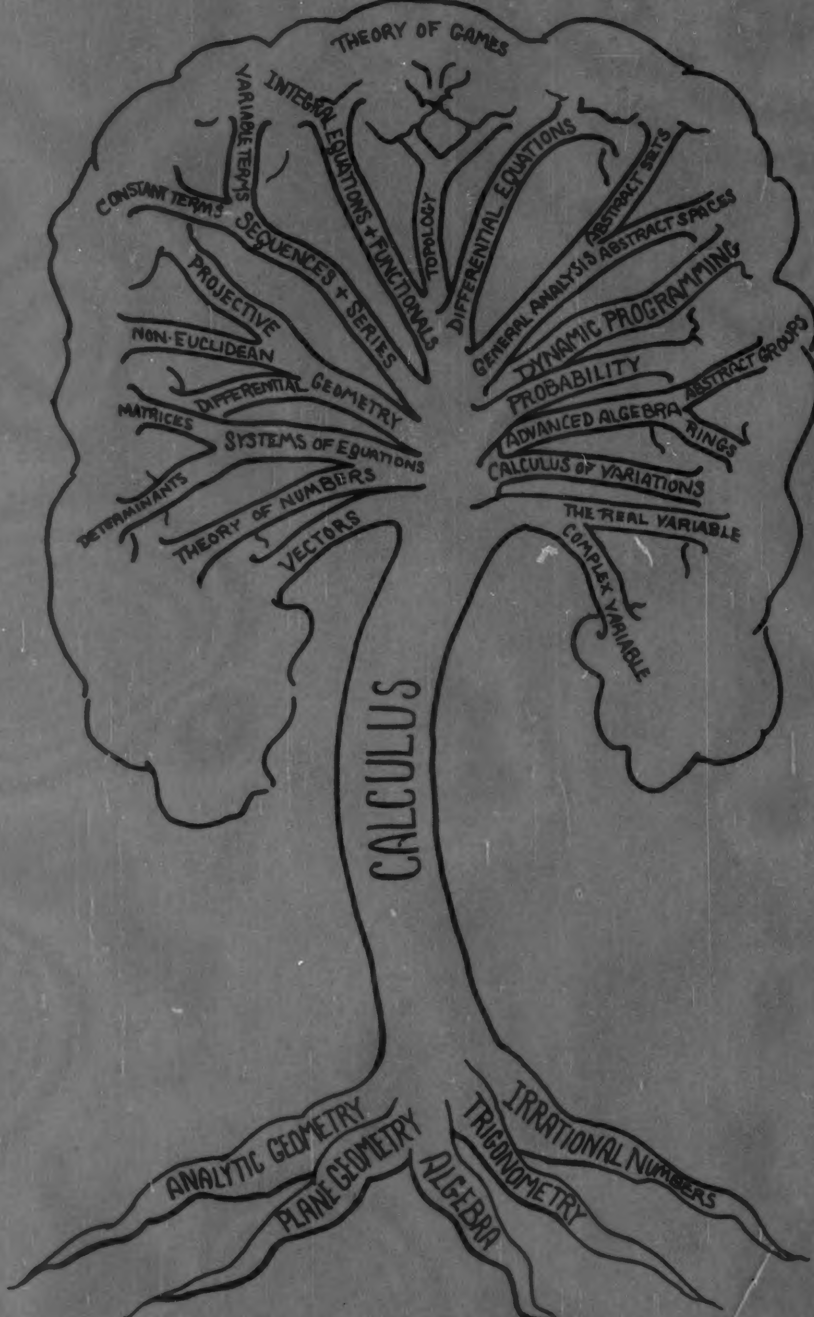
CONIC REFLECTIONS

I fear, ellipse, thy sheltered mind
Is just a modicum confined.
We can predict how it will wind
In bounded oscillations.

Thy mind, hyperbola, has sweep.
But when its thoughts are bold and deep
It may in schizophrenia leap
And break communications.

Parabola, thy mental state
Is one which we should emulate.
Though ranging wide, avoid the fate
Of babbling self-negations.

Marlow Sholander



See page 123.

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